Chapter 10

Casimir operator and character formula

In this chapter, we assume that the matrix $A$ is symmetrisable. We also fix an invariant bilinear for $(\ , \ )$ on $\mathfrak{g}(A)$. The existence of this bilinear form will ensure the existence of the Casimir operator which is an important tool in the character formula. Character formula is true for any Kac-Moody Lie algebra but one requires the construction of Kac-Moody group and the use of geometric arguments to prove it.

10.1 Casimir operator

10.1.1 Some formulas

Fix a root $\alpha \in \Delta_+$ and choose basis $(e^k_{\alpha})$ and $(e^k_{-\alpha})$ of the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ such that $(e^k_{\alpha}, e^l_{-\alpha}) = \delta_{k,l}$ for all $k$ and $l$. If $\alpha = \alpha_i$ is simple then there exists an unique vector $e^k_{\alpha_i}$ and we take $e^k_{\alpha_i} = e_i$. In that case because $(e_i, f_i) = \varepsilon_i$ we have $e^k_{-\alpha_i} = \frac{1}{\varepsilon_i} f_i$.

Lemma 10.1.1 For all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, we have:

$$(x, y) = \sum_k (x, e^k_{-\alpha})(y, e^k_{\alpha}).$$

Proof : Write $x = \sum_k (x, e^k_{-\alpha}) e^k_{\alpha}$ and $y = \sum_k (y, e^k_{\alpha}) e^k_{-\alpha}$, the result follows. □

Lemma 10.1.2 Let $\alpha$ and $\beta$ in $\Delta$ and let $z \in \mathfrak{g}_{\beta - \alpha}$, then we have in $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$:

$$\sum_s e^s_{-\alpha} \otimes [z, e^s_{\alpha}] = \sum_s [e^s_{-\beta}, z] \otimes e^s_{\beta}.$$ 

Proof : Define a non degenerate bilinear form on $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$ by $(x \otimes y, z \otimes t) = (x, z)(y, t)$. To prove the result, it is enough to prove that the equality holds after taking the bilinear form with $e^u_{\alpha} \otimes e^v_{-\beta}$ for all $u$ and $v$. This gives

$$\left( \sum_s e^s_{-\alpha} \otimes [z, e^s_{\alpha}], e^u_{\alpha} \otimes e^v_{-\beta} \right) = \sum_s \delta_{s,u}([z, e^s_{\alpha}], e^v_{-\beta}) = (e^u_{\alpha}, [e^v_{-\beta}, z])$$

and

$$\left( \sum_s [e^s_{-\beta}, z] \otimes e^s_{\alpha}, e^u_{\alpha} \otimes e^v_{-\beta} \right) = \sum_s \delta_{s,v}(e^u_{\alpha}, e^s_{-\beta}, z) = (e^u_{\alpha}, [e^v_{-\beta}, z]).$$
The result follows. \hfill \Box

**Corollary 10.1.3** With the notation of the previous lemma we have the formulas:

\[
\sum_s [e^{s}_{-\alpha}, [z, e^{s}_\alpha]] = - \sum_s [[z, e^{s}_{-\beta}], e^{s}_\beta] \text{ in } g(A),
\]

\[
\sum_s e^{s}_{-\alpha} [z, e^{s}_\alpha] = - \sum_s [z, e^{s}_{-\beta}] e^{s}_\beta \text{ in } U(g(A)),
\]

**Proof:** Apply the previous lemma and the maps from \( g(A) \otimes g(A) \) to \( g(A) \) and \( U(g(A)) \) respectively given by \((x, y) \mapsto [x, y]\) and \((x, y) \mapsto xy\). \hfill \Box

**Remark 10.1.4** Remark that the previous lemma and corollary are still true if one of the element \( \alpha \), \( \beta \) or \( \beta - \alpha \) is not a root.

### 10.1.2 Casimir operator

**Definition 10.1.5** We define a special element \( \rho \) in \( \mathfrak{h}^* \) as follows: take \( \rho \) to be a solution of the equations \( \langle \rho, \alpha_i^\vee \rangle = 1 \) for all index \( i \). In fact \( \rho \) is uniquely determined only in the finite type case. In that case it is given by half the sum of the positive roots:

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.
\]

In general we take for \( \rho \) any solution of these equations.

**Fact 10.1.6** It follows from the formula \( \langle \rho, \alpha_i^\vee \rangle = \frac{2(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} \) that \( (\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) \).

Choose as before a basis \( (e^s_\alpha) \) of \( g_\alpha \) and let \( (e^s_{-\alpha}) \) be its dual basis in \( g_{-\alpha} \).

**Definition 10.1.7** We define the operator \( \Omega_0 \) on any object \( V \) of \( \mathcal{O} \) by

\[
\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_s e^{s}_{-\alpha} e^{s}_\alpha.
\]

Remark that this is well defined since because \( V \) is in \( \mathcal{O} \), for any vector \( v \in V \), there is a finite number of positive root \( \alpha \) such that \( g_\alpha(v) \neq 0 \). Choose also \( (u_k) \) and \( (u^k) \) dual bases of \( \mathfrak{h} \).

**Definition 10.1.8** We define the Casimir operator \( \Omega \) on any object \( V \) of \( \mathcal{O} \) by

\[
\Omega = 2\nu^{-1}(\rho) + \sum_k u_k u^k + \Omega_0.
\]

**Lemma 10.1.9** This definitions of \( \Omega_0 \) and \( \Omega \) do not depend on the choice of the dual basis. Therefore definition of \( \Omega \) does only depend on the choice of \( \rho \) (and of the invariant bilinear form \( (\ , \ ) \)).

**Proof:** We denote \( \mathfrak{h} = g_0 \) and \( \alpha \) will be a root or 0. We identify \( g_{-\alpha} \) with \( g_\alpha^* \) thanks to the bilinear form. With this identification, the element \( \sum_s e^{s}_{-\alpha} \otimes e^{s}_\alpha \in g_{-\alpha} \otimes g_\alpha = g_\alpha^* \otimes g_\alpha \) correspond to the identity. In particular, it does not depend on the choice of the base. \hfill \Box
Remark 10.1.10 The operators \(\Omega_0\) and \(\Omega\) live in a completion \(\hat{U}(g(A))\) of the enveloping algebra \(U(g(A))\). Indeed, let \(U_d(n)\) be the subspace of \(U(n)\) formed by the degree \(d\) homogeneous elements (the grading is given by the height) and define

\[
\hat{U}(g(A)) = \prod_{d \geq 0} U(n_\cdot) \otimes U(h) \otimes U_d(n).
\]

There a natural product given by

\[
\sum_{d \geq 0} x_d \cdot \sum_{m \geq 0} y_m = \sum_{k} \sum_{d,m \geq 0} (x_d y_m)_k
\]

where for \(x_d\) and \(y_d\) in \(U(n_-) \otimes U(h) \otimes U_d(n_+ )\) the element \((x_d y_m)_k d\) denotes the component of \(x_d y_m\) in the factor \(U(n_-) \otimes U(h) \otimes U_k(n_+ )\).

Let us prove the main result on the Casimir operator:

Theorem 10.1.11 The action of the Casimir operator \(\Omega\) on any module \(V\) of the category \(\mathcal{O}\) commutes with the action of \(g(A)\). In other words \(\Omega\) lies in \(Z(\hat{U}(g(A)))\) the center of the completed enveloping algebra.

Proof: Since the centraliser \(Z(\Omega)\) of \(\Omega\) is a subalgebra, it suffices to prove that \(\Omega\) commute with the generators i.e. with the elements \(e_i, f_i\) and \(h\) for \(h \in h\). But the weight of \(\Omega_0\) is zero as well as the other components of \(\Omega\) proving the commutation with \(h\) for all \(h \in h\). We start with the following:

Lemma 10.1.12 For \(x \in g_\alpha\), we have in \(U(g(A))\):

\[
\sum_{k} [u^k u_k, x] = x((\alpha, \alpha) + 2\nu^{-1}(\alpha)).
\]

Proof: We first compute \(u^k u_k x = u^k [u_k, x] + u^k x u_k = \langle \alpha, u_k \rangle u^k x + u^k x u_k\) and \(x u_k u_k = [x, u^k] u_k + u^k x u_k = -\langle \alpha, u^k \rangle x u_k + u^k x u_k\). In particular we get

\[
\sum_{k} [u^k u_k, x] = \sum_{k} \langle \alpha, u_k \rangle u^k x + \sum_{k} \langle \alpha, u^k \rangle x u_k.
\]

But \(u^k x = [u^k, x] + x u_k = \langle \alpha, u^k \rangle x + x u^k\) thus we have

\[
\sum_{k} [u^k u_k, x] = \sum_{k} \langle \alpha, u_k \rangle \langle \alpha, u^k \rangle x + x \sum_{k} \langle \alpha, u_k \rangle u^k + \langle \alpha, u^k \rangle u_k.
\]

But recall the formulas \(\lambda = \sum_k \langle \lambda, u^k \rangle \nu(u_k) = \sum_k \langle \lambda, u_k \rangle \nu(u^k)\) giving \((\lambda, \mu) = \sum_k \langle \lambda, u^k \rangle \langle \mu, u_k \rangle\). The result follows.

We compute the second part of the Casimir operator:

Lemma 10.1.13 The following formula holds in \(\hat{U}(g(A))\):

\[
[\Omega_0, e_i] = -2e_i((\alpha_i, \alpha_i) + \nu^{-1}(\alpha_i)).
\]
Proof: In the following sums, we may regard the elements $\alpha$ as positive roots or elements in $Q_+$. In all the terms of the following equalities, this will be the same because by convention we take $e^\alpha = e^{-\alpha} = 0$ if $\alpha$ is not a root and because of Remark 10.1.4. Let us now compute

$$[\Omega_0, e_i] = 2 \sum_{\alpha \in \Delta_+} \sum_s [e^\alpha e^s, e_i] = 2 \sum_{\alpha \in \Delta_+} \sum_s ([e^s, e_i]e^\alpha + e^\alpha [e^s, e_i])$$

$$= 2[e^{-\alpha}, e_i]e_i + 2 \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \left( \sum_s [e^\alpha e_i]e^s + \sum_s [e_i, e^\alpha e^s - e^s e^\alpha]e^s e_i \right).$$

The last equality comes from Lemma 10.1.3. Furthermore, the last sum is equal to the same sum but with indices of summation $\alpha \in Q_+$. Furthermore, if $\alpha$ is a root not containing $\alpha_i$ in its support, then $e^\alpha$ is a linear combination of brackets $[I, \ldots, [I_i]]$ with all indices $i_i$ different from $i$. In particular the Lie bracket $[[f_i, \ldots, [f_{i_{p-1}}, f_{i_p}]], e_i]$ vanishes. This implies that the same roots appear in the two terms of the second expression which necessary vanishes. We end up with

$$[\Omega_0, e_i] = 2[e^{-\alpha_i}, e_i]e_i = -\frac{2}{e_i^{\alpha_i}}e_i = -\frac{2}{e_i^{\alpha_i}}(\alpha_i^{\alpha_i} e_i) - \frac{2}{e_i^{\alpha_i}}e_i = -2(\alpha_i, e_i) - 2\nu^{-1}(\alpha_i).$$

The apparition of $e_i$ comes from the fact that the dual of $e_i$ is $\frac{1}{e_i} f_i$ (see the definition of the invariant bilinear form in Theorem 6.2.5).

Now Lemma 10.1.12 gives us

$$\sum_k [x^k u_k, e_i] = e_i((\alpha_i, \alpha_i) + 2\nu^{-1}(\alpha_i)).$$

Putting all these formulas together we get:

$$[\Omega, e_i] = [2\nu^{-1}(\rho), e_i] + (\alpha_i, \alpha_i)e_i + 2\nu^{-1}(\alpha_i) - 2(\alpha_i, \alpha_i)e_i - 2\nu^{-1}(\alpha_i)$$

$$= [2\nu^{-1}(\rho), e_i] - (\alpha_i, \alpha_i)e_i.$$}

But $[2\nu^{-1}(\rho), e_i] = 2(\alpha_i, \nu^{-1}(\rho))e_i = 2(\rho, \alpha_i)e_i = (\alpha_i, \alpha_i)e_i$. We get the desired formula

$$[\Omega, e_i] = 0.$$

The same proof works with $f_i$ and the result follows.

**Corollary 10.1.14** (i) For any $\lambda \in \mathfrak{h}^*$, we have $\Omega_{M(\lambda)} = (|\lambda + \rho|^2 - |\rho|^2)\operatorname{Id}_{M(\lambda)}$.

(ii) In particular, for any subquotient $V$ of $M(\lambda)$ we have $\Omega|_V = (|\lambda + \rho|^2 - |\rho|^2)\operatorname{Id}_V$.

**Proof:** We will consider, as in Proposition 9.3.2, the Verma module $M(\lambda)$ as the quotient:

$$M(\lambda) = U(\mathfrak{g}(A))/I(\lambda).$$

It is generated by 1 and because of the previous theorem, it suffices to show that $\Omega(1) = (|\lambda + \rho|^2 - |\rho|^2)1$. But we have $n_+(1) = 0$ and $h(1) = \langle \lambda, h \rangle 1$ for $h \in \mathfrak{h}$. This give the formula

$$\Omega(1) = (2\nu^{-1}(\rho), \lambda)1 + \sum_k \langle \lambda, u^k \rangle \langle \lambda, u_k \rangle 1$$

$$= ((2\rho, \lambda) + (\lambda, \lambda))1$$

$$= ((\rho + \lambda, \rho + \lambda) - (\rho, \rho))1.$$
10.2 Character formula

Let $\rho$ be any solution of the system $\langle \rho, \alpha_i \rangle = 1$ for all $i$ (such a $\rho$ is unique only is the finite case). We prove in this section the following:

**Theorem 10.2.1** Let $L$ be an integrable highest weight module. Then we have:

$$
\text{Ch}(L(\lambda)) = \frac{\sum w \in W \epsilon(w) e(w(\lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}_\alpha}}.
$$

**Corollary 10.2.2** Let $L(\lambda)$ be an irreducible and integrable highest weight module. Then we have:

$$
\text{Ch}(L(\lambda)) = \frac{\sum w \in W \epsilon(w) e(w(\lambda + \rho))}{\sum w \in W \epsilon(w) e(w(\rho))}.
$$

**Proof:** Apply the theorem to $L(0)$ which is the trivial representation. Its character is the unit and we get the so-called denominator identity:

$$
\sum_{w \in W} \epsilon(w) e(w(\rho) - \rho) = \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}_\alpha}.
$$

The result follows. $\Box$

To prove Theorem 10.2.1 we need some lemmas. First remark that the Weyl groups acts naturally on the algebra $E$ by $w(e(\lambda)) = e(w(\lambda))$.

**Lemma 10.2.3** Let $R$ be the element (the denominator) $\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}_\alpha}$ in $E$. Then we have

$$
w(e(\rho)R) = e(\rho)e(\rho)R.
$$

**Proof:** It suffices to prove this for simple reflections. We have

$$
s_i(e(\rho)R) = e(\rho - \alpha_i) \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(s_i(-\alpha))^{\text{mult}_\alpha}
$$

$$
= e(\rho - \alpha_i)(1 - e(-\alpha_i))^{\text{mult}_\alpha} \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(s_i(-\alpha))^{\text{mult}_\alpha}
$$

$$
= e(\rho)e(-\alpha_i)(1 - e(-\alpha_i)) \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^{\text{mult}_\alpha}
$$

$$
= -e(\rho)R.
$$

The result follows. $\Box$

**Lemma 10.2.4** Let $V$ be a highest weight module of highest weight $\lambda$, then

$$
\text{Ch}V = \sum_{\mu \leq \lambda, \ |\mu + \rho| = |\lambda + \rho|} c_\mu \text{Ch}M(\mu)
$$

where $c_\mu \in \mathbb{Z}$ and $c_\lambda = 1$. 

Proof: It is sufficient to prove this result for $L(\lambda)$ because of Proposition 9.7.4. Now we consider the equation given by Proposition 9.7.4

$$\text{Ch}M(\mu) = \sum_{\nu \in h^*} [M(\mu) : L(\nu)] \text{Ch}L(\nu).$$

But Corollary 10.1.14 tells us that the action of the Casimir element on $M(\mu)$ is $(|\mu + \rho|^2 - |\rho|^2)\text{Id}$ and the same for all its subquotient so in particular on $L(\nu)$ if the multiplicity is non zero. But the action of the Casimir on $L(\nu)$ is $(|\nu + \rho|^2 - |\rho|^2)\text{Id}$ so this implies that $|\mu + \rho|^2 = |\nu + \rho|^2$.

Consider the set $S(\mu) = \{\nu \in h^* / \nu \leq \mu \text{ and } |\nu + \rho|^2 = |\mu + \rho|^2\}$. For any $\mu$ we have an equation

$$\text{Ch}M(\mu) = \sum_{\nu \in S(\mu)} [M(\mu) : L(\nu)] \text{Ch}L(\nu).$$

with $[M(\mu) : L(\mu)] = 1$. This system is triangular and in particular considering this system for $\mu \in S(\lambda)$ we get the result by inverting the system. \(\square\)

**Lemma 10.2.5** Let $\lambda \in h^*$ be such that $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ for all index $i$. Then for any $\nu \in h^*$ such that

- $\nu \leq \lambda + \rho$,
- $(\nu, \nu) = (\lambda + \rho, \lambda + \rho)$,
- $(\nu, \alpha_i^\vee) \geq 0$ for all $i$,

we have $\nu = \lambda + \rho$.

**Proof:** Write $\nu = \lambda + \rho - \sum_i a_i \alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$. We have

$$(\nu, \nu) = (\lambda + \rho, \lambda + \rho) - (\lambda + \rho, \sum_i a_i \alpha_i) = (\nu, \sum_i a_i \alpha_i).$$

We thus have $((\lambda + \rho, \sum_i a_i \alpha_i) + (\nu, \sum_i a_i \alpha_i))$. But the second term is non positive by hypothesis and the first one is non negative (recall that $(\rho, \alpha_i) = \frac{1}{2} (\alpha_i, \alpha_i) > 0$. This equality is possible if and only if all the $a_i$ vanish. \(\square\)

We prove the character formula.

**Proof:** There exist integers $d_\mu$ with $d_\lambda = 1$ such that:

$$\text{Ch}L = \sum_{\mu \in S(\lambda)} d_\mu \text{Ch}M(\mu).$$

By multiplying by $e(\rho) R$ and thanks to Proposition 9.7.5 we get

$$e(\rho) R \cdot \text{Ch}L = \sum_{\mu \in S(\lambda)} d_\mu e(\mu + \rho).$$

But now recall that because $L$ is integrable, its character is $W$-invariant. This together with the $W$-anti-invariance of $e(\rho) R$ gives:

$$d_\mu = e(w) d_{w(\mu + \rho) - \rho}.$$

Fix $\mu$ with $d_\mu \neq 0$. Then for any $w \in W$, we have $d_{w(\mu + \rho) - \rho} \neq 0$ thus $w(\mu + \rho) - \rho \leq \lambda$. Take $v \in W$ such that $\text{ht}(\lambda - (v(\mu + \rho) - \rho))$ is minimal. Set $\nu = v(\mu + \rho)$.  

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We have $\langle \nu, \alpha_i^\vee \rangle \geq 0$. Indeed, the condition on the height imply the inequality $\text{ht}(\lambda - s_i v(\mu + \rho) + \rho) \geq \text{ht}(\lambda - v(\mu + \rho) + \rho)$. But this implies that $\langle v(\mu + \rho), \alpha_i^\vee \rangle \geq 0$.

We have $\nu = v(\mu + \rho)$ and $\langle \nu, \nu \rangle = (\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$ because $\mu \in S(\lambda)$. Now the previous lemma implies that $\nu = \lambda + \rho$. In particular, for any $\mu$ with $d_{\mu} \neq 0$, we have that there exists a $w \in W$ (here $w = v^{-1}$) such that $\mu = w(\lambda + \rho) - \rho$ and

$$d_{\mu} = \epsilon(v) d_{w(\mu + \rho) - \rho} = \epsilon(v) d_{\lambda} = \epsilon(v) = \epsilon(w).$$

We thus have

$$e(\rho) R \cdot \text{Ch}(L) = \sum_{w \in W} \epsilon(w) e(w(\lambda + \rho))$$

$$R \cdot \text{Ch}(L) = \sum_{w \in W} \epsilon(w) e(w(\lambda + \rho) - \rho)$$

and the result follows. \qed

**Corollary 10.2.6** An integrable highest weight module $L$ is irreducible.

**Proof:** Indeed, the module $L$ and its irreducible quotient have the same character. \qed
Bibliography


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