Chapter 11

Root systems

This chapter is independent of the previous chapters. We define and study the root systems and give a classification of all root systems. In this chapter, all vector spaces will be considered over \( \mathbb{R} \) the field of real numbers. All vector spaces will be finite dimensional.

11.1 Definition

Definition 11.1.1 Let \( V \) be a finite dimensional vector space and \( \alpha \in V \) a nonzero vector. A symmetry with vector \( \alpha \) is an automorphism \( s \) of \( V \) such that

- \( s(\alpha) = -\alpha \)
- the set \( H = \{ \beta \in V / s(\beta) = \beta \} \) of fixed elements is an hyperplane of \( V \).

We have the following fact where we use the notation \( \langle f, v \rangle = f(v) \) for \( f \in V^\vee \) and \( v \in V \).

Fact 11.1.2 Let \( s \) be a symmetry with vector \( \alpha \).

(i) The space \( H = \{ \beta \in V / s(\beta) = \beta \} \) is a complement for \( \mathbb{R} \alpha \) in \( V \).

(ii) The element \( s \) has order 2.

(iii) There is a unique element \( \alpha^\vee \in V^\vee \) such that \( \langle \alpha^\vee, H \rangle = 0 \) and \( \langle \alpha^\vee, \alpha \rangle = 2 \). We have

\[
s(v) = v - \langle \alpha^\vee, v \rangle \alpha,
\]

(iv) If \( \alpha \) and \( \alpha^\vee \) are elements in \( V \) and \( V^\vee \) such that \( \langle \alpha^\vee, \alpha \rangle = 2 \), then the map \( s \) defined by \( s(v) = v - \langle \alpha^\vee, v \rangle \alpha \) is a symmetry with vector \( \alpha \).

Proof. (i) The space \( H \) is of codimension 1 and does not contain \( \alpha \), the result follows.

(ii) The order of \( s \) on \( H \) is 1 and 2 on \( \mathbb{R} \alpha \), the result follows from (i).

(iii) The uniqueness is clear because \( H \) and \( \mathbb{R} \alpha \) are in direct sum. Furthermore, for \( v \in V \), write \( v = h + \lambda \alpha \), we have \( s(v) = h - \lambda \alpha \) and \( v - \langle \alpha^\vee, v \rangle \alpha = h + \lambda \alpha - 2\lambda \alpha = h - \lambda \alpha \) and the formula follows.

(iv) Let \( H \) be the kernel of \( \alpha^\vee \). It is an hyperplane and for \( h \in H \), we have \( s(h) = h \). Furthermore we have the equality \( s(\alpha) = -\alpha \) proving the result. \( \square \)

Lemma 11.1.3 Let \( \alpha \) be a nonzero element in \( V \) and \( R \) be a finite subset of \( V \) which spans \( V \). Then there is at most one symmetry with vector \( \alpha \) leaving \( R \) invariant.
Proof. Let \( s \) and \( s' \) be two such symmetries and let \( u = s \circ s' \).

On the one hand, we have \( u(\alpha) = \alpha \) and \( u \) induces the identity on the quotient \( V/\mathbb{R} \alpha \). This proves that the eigenvalues of \( u \) are all equal to 1.

On the other hand, we have \( u(R) = R \) therefore \( u \) induces a permutation of \( R \) and therefore, there exists an integer \( n \) such that \( (u|_R)^n = \text{Id}_R \). But because \( R \) spans \( V \) we get \( u^n = \text{Id}_V \). In particular \( u \) is semisimple. As is has only 1 as eigenvalue we get \( u = \text{Id}_V \) and \( s = s' \). \( \square \)

**Definition 11.1.4** subset \( R \) of a vector space \( V \) is called a root system in \( V \) if the following conditions are satisfied:

1. \( R \) is finite, spans \( V \) and does not contain 0;
2. for each \( \alpha \in R \), there exists a symmetry \( s_\alpha \) with vector \( \alpha \) leaving \( R \) invariant;
3. for each \( \alpha \) and \( \beta \) in \( R \), the vector \( s_\alpha(\beta) - \beta \) is an integer multiple of \( \alpha \) (i.e. \( \langle \alpha^\vee, \beta \rangle \in \mathbb{Z} \)).

Remark that the symmetry \( s_\alpha \) is unique by the above Lemma. The dimension of \( V \) is called the rank of the root system and the elements \( \alpha \) in \( R \) are called the roots of the root system. The element \( \alpha^\vee \) is called the dual root or inverse root of \( \alpha \).

**Remark 11.1.5** By (2), we have for \( \alpha \) in \( R \) that \( -\alpha = s_\alpha(\alpha) \in R \).

**Definition 11.1.6** A root system is called reduced if the intersection of \( R \) with \( \mathbb{R} \alpha \) for \( \alpha \in R \) is the set \( \{\alpha, -\alpha\} \).

**Fact 11.1.7** Let \( R \) be a nonreduced root system and \( \alpha \in R \) such that \( R \cap \mathbb{R} \alpha \) contains more than the two roots \( \{-\alpha, \alpha\} \), then we have

\[
R \cap \mathbb{R} \alpha = \{-2\alpha, -\alpha, \alpha, 2\alpha\}, \text{ or } R \cap \mathbb{R} \alpha = \{-\alpha, -\frac{1}{2} \alpha, \frac{1}{2} \alpha, \alpha\}.
\]

**Proof.** By taking for \( \alpha \) the root with biggest coefficient in \( R \cap \mathbb{R} \alpha \), we may assume that any other root \( \beta \in \mathbb{R} \cap \mathbb{R} \alpha \) is of the form \( t\alpha \) with \( 0 < t < 1 \). Applying point (3) of the definition, we have \( s_\alpha(\beta) - \beta = -t\alpha - t\alpha = -2t\alpha \in \mathbb{Z} \alpha \). Therefore we have \( t = \frac{1}{2} \) and the result follows. \( \square \)

**Example 11.1.8** (i) The only reduced root system of rank 1 is \( \{-\alpha, \alpha\} \) and is called of type \( A_1 \).

(ii) The only nonreduced root system of rank 1 is \( \{-2\alpha, -\alpha, \alpha, 2\alpha\} \).

(iii) The following subsets in \( \mathbb{R}^2 \) are root systems:

- \( \{-\beta, -\alpha, \alpha, \beta\} \) with \( \alpha = (1, 0) \) and \( \beta = (0, 1) \). Root system of type \( A_1 \times A_1 \).
- \( \{-\alpha - \beta, -\alpha, \alpha, \beta, \alpha + \beta\} \) with \( \alpha = (1, 0) \) and \( \beta = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \). Root system of type \( A_2 \).
- \( \{-2\alpha - \beta, -\alpha - \beta, -\alpha, \alpha, \beta, \alpha + \beta, 2\alpha + \beta\} \) with \( \alpha = (1, 0) \) and \( \beta = (-1, 1) \). Root system of type \( B_2 = C_2 \).
- \( \{-3\alpha - 2\beta, -3\alpha - \beta, -2\alpha - \beta, -\alpha - \beta, -\alpha, \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\} \) with \( \alpha = (1, 0) \) and \( \beta = (-\frac{3}{2}, \frac{\sqrt{3}}{2}) \). Root system of type \( G_2 \).
11.2 Weyl group

Definition 11.2.1 Let $R$ be a root system in the $\mathbb{R}$-vector space $V$.

(i) The Weyl group of $R$ is the subgroup of $\text{GL}(V)$ generated by the symmetric $s_\alpha$ for $\alpha \in R$, we denote it by $W(R)$.

(ii) The group of automorphisms of $R$ is the subgroup of all elements in $\text{GL}(V)$ preserving $R$, we denote it by $\text{Aut}(R)$.

Fact 11.2.2 The Weyl group $W(R)$ is a normal subgroup of $\text{Aut}(R)$ and both are finite.

Proof. The two groups are contained in the group of permutation of $R$ (the map $\text{Aut}(R) \to \mathcal{S}(R)$ is injective because $R$ generates $V$). Furthermore, if $u \in \text{Aut}(R)$, then $us_\alpha u^{-1} = s_{u(\alpha)}$ for all $\alpha \in R$ therefore $W(R)$ is normal in $\text{Aut}(R)$. \hfill $\square$

Example 11.2.3 (i) When $R$ is a reduced root system of rank 2 as in the previous example, then the Weyl group is isomorphic to the dihedral group of order $2n$ (which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$) with $n = 2$ for type $A_1 \times A_1$, $n = 3$ for $A_2$, $n = 4$ for $B_2$ and $n = 6$ for $G_2$.

(ii) For the automorphism group $\text{Aut}(R)$ of the previous example, we have $|\text{Aut}(R) : W(R)| = 2$ for type $A_1 \times A_1$ and $A_2$ and $\text{Aut}(R) = W(R)$ in the last two cases.

11.3 Invariant bilinear form

Definition 11.3.1 Let $B$ be a bilinear form on $V$, an element $u \in \text{GL}(V)$ preserves $B$ if for all $v$ and $v'$ in $V$, we have $B(u(v), u(v')) = B(v, v')$. The subgroup of all elements preserving $B$ is called the orthogonal group associated to $B$ and denoted by $O(V, B)$.

Definition 11.3.2 A bilinear form $B$ on $V$ is called invariant under a subgroup $G \subset \text{GL}(V)$ if we have $G \subset O(V, B)$.

Proposition 11.3.3 Let $R$ be a root system, there exists a positive definite symmetric bilinear form $(\, , \, )$ on $V$ which is invariant under the Weyl group $W(R)$.

Proof. Let $B$ be any positive definite bilinear form on $V$. We define the following symmetric bilinear form:

$$(v, v') = \frac{1}{|W(R)|} \sum_{u \in W(R)} B(u(v), u(v')).$$

We have that $(\, , \, )$ is positive definite and invariant under the Weyl group $W(R)$. \hfill $\square$

From now on, we fix a positive definite $W(R)$-invariant bilinear form $(\, , \, )$ on $V$ which has therefore the structure of an Euclidean space. We denote by $O(V)$ the group $O(V, (\, , \, ))$. We have the inclusion $W(R) \subset O(V)$.

Fact 11.3.4 For $\alpha \in R$ and $v \in V$ we have the formula

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha.$$

Equivalently, identifying $V$ with $V^\vee$ using $(\, , \, )$ we have the equality

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$
Proof. Let $v \in V$ such that $s_\alpha(v) = v$, we have $(v, \alpha) = (s_\alpha(v), s_\alpha(\alpha)) = (-\alpha, v)$ therefore $(v, \alpha) = 0$. For an element $v \in V$, we write $v = \lambda \alpha + v_0$ with $v_0 \in \alpha^\perp$ (the orthogonal being taken for the form $(\ , \ )$). We have $s_\alpha(v) = -\lambda \alpha + v_0$ and $v - 2\frac{(\alpha, \alpha)}{(\alpha, \alpha)}\alpha = v - 2\lambda \alpha = s_\alpha(v)$.

This means that for $v \in V$, we have $\langle \alpha^\vee, v \rangle = 2\frac{(\alpha, v)}{(\alpha, \alpha)} = \left(\frac{2\alpha}{(\alpha, \alpha)}, v\right)$ giving the second assertion. \qed

11.4 Dual root system

Proposition 11.4.1 Let $R$ be a root system and denote by $R^\vee$ the set of all dual roots $\alpha^\vee$ for $\alpha \in R$.

(i) The set $R^\vee$ in $V^\vee$ is a root system.

(ii) We have $(\alpha^\vee)^\vee = \alpha$ and $(R^\vee)^\vee = R$.

Proof. Let us check the three axioms of roots systems on $R^\vee$.

(1) The set $R^\vee$ is in bijection with $R$ and is therefore finite. Furthermore, for $\alpha \in R$, the symmetry $s_\alpha$ is not the identity (because $s_\alpha(\alpha) = -\alpha$) therefore $\alpha^\vee$ is not trivial. Finally, the set $R^\vee$ spans $V^\vee$. Indeed, if not there would be a non-zero vector $v \in V$ such that $\alpha^\vee(v) = 0$ for all $\alpha \in R$. This would imply $v \in R^\perp$ but $R$ spans $V$ thus we have $R^\perp = V^\perp = 0$ a contradiction.

(2) Let us define $s_\alpha^{\alpha^\vee}(f) = f - (f, \alpha)\alpha^\vee$. This is a symmetry in $V^\vee$ and $s_\alpha^{\alpha^\vee}(\alpha^\vee) = -\alpha^\vee$ therefore it has vector $\alpha^\vee$. We have $s_\alpha^{\alpha^\vee} = ts_\alpha$. Indeed, by definition of the transpose, for any $f \in V^\vee$ and $v \in V$, we have $(s_\alpha)(f)(v) = f \circ s_\alpha(v) = f(v - \langle \alpha, v \rangle \alpha) = f(v) - \langle \alpha^\vee, v \rangle f(\alpha) = (f - (f, \alpha)\alpha^\vee)(v) = s_\alpha^{\alpha^\vee}(f)(v)$.

In particular, for $\alpha, \beta \in R$ and $v \in V$, we have

$$\langle s_\alpha^{\alpha^\vee}(\beta^\vee), v \rangle = \langle \beta^\vee, s_\alpha(v) \rangle = \frac{2(\beta, s_\alpha(v))}{(\beta, \beta)} = \frac{2(s_\alpha(\beta), v)}{(s_\alpha(\beta), s_\alpha(\beta))} = \langle s_\alpha(\beta)^\vee, v \rangle$$

therefore $s_\alpha^{\alpha^\vee}(\beta^\vee) = s_\alpha(\beta)^\vee$ and $s_\alpha^{\alpha^\vee}$ preserves $R^\vee$.

We see from this point that $(\alpha^\vee)^\vee = \alpha$ and that $(R^\vee)^\vee = R$.

(3) This condition is equivalent to the condition $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha$ and $\beta$ in $R$. But we have $\langle \alpha^\vee, \beta \rangle = \langle \alpha^\vee, (\beta^\vee)^\vee \rangle$ and point (3) follows. \qed

Definition 11.4.2 The root system $R^\vee$ is called the dual root system.

Proposition 11.4.3 The Weyl group $W(R^\vee)$ is isomorphic to the Weyl group $W(R)$.

More precisely, $\Phi : \text{GL}(V) \to \text{GL}(V^\vee)$ be the isomorphism defined by $\Phi(u) = t_u^{-1}$. Then $\Phi$ restricts to an isomorphism from $W(R)$ to $W(R^\vee)$.

Proof. We have seen that $t_s_\alpha = s_\alpha^{\alpha^\vee}$ therefore $\Phi(s_\alpha) = s_\alpha^{\alpha^\vee}$. Furthermore, because $\Phi$ is involutive we get the result. \qed

11.5 Relative position of two roots

Let $R \subset V$ be a root system. From now on we fix a positive definite bilinear form $(\ , \ )$ on $V$ which is invariant under the Weyl group $W = W(R)$. This defines an Euclidian structure on $V$. We want in this section to describe all possibilities for the relative position of two roots.

Let $\alpha$ and $\beta$ be two roots and let us denote by $|\alpha|$ the length of $\alpha$ (i.e. $|\alpha| = \sqrt{(\alpha, \alpha)}$). Let us denote by $\phi$ the angle between the lines generated by $\alpha$ and $\beta$. Remark that if $\alpha$ and $\beta$ are colinear, we already know they relative position.
**Proposition 11.5.1** If $\alpha$ and $\beta$ are non colinear, then there are 7 possibilities (up to transposition of $\alpha$ and $\beta$):

- $\langle \alpha^\vee, \beta \rangle = 0, \langle \beta^\vee, \alpha \rangle = 0$ and $\phi = \pi/2$;
- $\langle \alpha^\vee, \beta \rangle = 1, \langle \beta^\vee, \alpha \rangle = 1$, $\phi = \pi/3$ and $|\beta| = |\alpha|$;
- $\langle \alpha^\vee, \beta \rangle = -1, \langle \beta^\vee, \alpha \rangle = -1$, $\phi = 2\pi/3$ and $|\beta| = |\alpha|$;
- $\langle \alpha^\vee, \beta \rangle = 2, \langle \beta^\vee, \alpha \rangle = 1$, $\phi = \pi/4$ and $|\beta| = \sqrt{2}|\alpha|$;
- $\langle \alpha^\vee, \beta \rangle = -2, \langle \beta^\vee, \alpha \rangle = -1$, $\phi = 3\pi/4$ and $|\beta| = \sqrt{2}|\alpha|$;
- $\langle \alpha^\vee, \beta \rangle = 3, \langle \beta^\vee, \alpha \rangle = 1$, $\phi = \pi/6$ and $|\beta| = \sqrt{3}|\alpha|$;
- $\langle \alpha^\vee, \beta \rangle = -3, \langle \beta^\vee, \alpha \rangle = -1$, $\phi = 5\pi/6$ and $|\beta| = \sqrt{3}|\alpha|$.

**Proof.** We have the equality $(\alpha, \beta) = |\alpha||\beta|\cos \phi$. In particular, we get

$$2\frac{|\beta|}{|\alpha|} \cos \phi = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$$

and we deduce that $4\cos^2 \phi = \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \in \mathbb{Z}$. We choose $\alpha$ and $\beta$ such that $|\alpha| \leq |\beta|$. This implies that $\langle \alpha^\vee, \beta \rangle \geq \langle \beta^\vee, \alpha \rangle$.

The previous relation gives in particular that we have $4\cos^2 \phi = 0; 1; 2; 3; 4$ and the corresponding values for $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$ are 0, 0; 1, 1 or $-1, -1$; 2, 1 or $-2, -1$; 3, 1 or $-3, -1$; 4, 1 or $-4, -1$. Remark that in the case $\cos^2 \phi = 1$, then $\phi = 0$ or $\pi$ and $\alpha$ and $\beta$ are colinear, so this case does not happen.

In the other cases, we have $\cos \phi = 0$: $\pm 1/2; \pm \sqrt{2}/2; \pm \sqrt{3}/2$ and therefore $\phi = \pi/2; \pi/3$ or $2\pi/3; \pi/4$ or $3\pi/4; \pi/6$ or $5\pi/6$ and the result follows. $\square$

**Proposition 11.5.2** Let $\alpha$ and $\beta$ be non colinear roots. If $\langle \alpha^\vee, \beta \rangle > 0$, then $\alpha - \beta$ is a root.

**Proof.** From the previous proposition, we get that $\langle \alpha^\vee, \beta \rangle = 1$ of $\langle \beta^\vee, \alpha \rangle = 1$. In the first case, we have $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha = \beta - \alpha$ therefore $\alpha - \beta = -s_\alpha(\beta) \in R$. In the second case, we have $\alpha - \beta = \alpha - \langle \beta^\vee, \alpha \rangle \beta = s_\beta(\alpha) \in R$. $\square$

### 11.6 System of simple roots

**Definition 11.6.1** A subset $S$ of $R$ is called a system of simple roots or a base of $R$ if the following two conditions are satisfied:

- $S$ is a basis for $V$;
- each root $\beta \in R$ can be written as a linear combination of roots in $S$ as $\beta = \sum_{\alpha \in S} a_\alpha \alpha$ with the alternative
  - $a_\alpha \geq 0$ for all $\alpha \in S$ or
  - $a_\alpha \leq 0$ for all $\alpha \in S$. 

Example 11.6.2 With the notation as in Example 11.1.8, the set \( S = \{ \alpha, \beta \} \) is a base for the different root systems.

Theorem 11.6.3 There exists a base.

Proof. We will prove a more precise statement. Let \( t \in V^\vee \) be an element such that \( \langle t, \alpha \rangle \neq 0 \) for all \( \alpha \in R \). This is possible because \( R \) is a finite set and \( \mathbb{R} \) an infinite field. Define the set
\[
R^+_t = \{ \alpha \in R \mid \langle t, \alpha \rangle > 0 \}.
\]
We have \( R = R^+_t \cup (-R^+_t) \).

Definition 11.6.4 An element \( \alpha \in R^+_t \) is called decomposable if there exists two root \( \beta \) and \( \gamma \) in \( R^+_t \) such that \( \alpha = \beta + \gamma \). If \( \alpha \) is not decomposable, it is called indecomposable. We denote by \( S_t \) the set of indecomposable elements in \( R^+_t \).

We will prove that \( S_t \) is a base for the root system and that any base for the root system is of the form \( S_t \).

Lemma 11.6.5 Any element \( \alpha \in R^+_t \) is a linear combination with non-negative integer coefficients of elements in \( S_t \).

Proof. Let \( I \) be the set of all elements \( \alpha \in R^+_t \) for which the above property is not satisfied. If \( I \) were non empty, then there exists an element \( \alpha \in I \) with \( \langle t, \alpha \rangle \) minimal. But if \( \alpha \) is not in \( S_t \) (otherwise \( \alpha \) can be written as a linear combination with non-negative integer coefficients of elements in \( S_t \)), therefore it is reducible and \( \alpha = \beta + \gamma \) with \( \beta, \gamma \in R^+_t \). We have \( \beta \in I \) or \( \gamma \in I \) (otherwise \( \alpha \in I \)). Say we have \( \beta \in I \), then we have \( \langle t, \beta \rangle = \langle t, \alpha \rangle - \langle t, \gamma \rangle < \langle t, \alpha \rangle \) a contradiction to the minimality of \( \langle t, \alpha \rangle \). \( \square \)

Lemma 11.6.6 We have \( (\alpha, \beta) \leq 0 \) for all \( \alpha \) and \( \beta \) in \( S_t \).

Proof. If \( (\alpha, \beta) > 0 \), then by Proposition 11.5.2 (remark that \( \alpha \) and \( \beta \) cannot be colinear otherwise one of them would not be in \( S_t \)) we know that \( \gamma = \alpha - \beta \) is a root. If \( \gamma \in R^+_t \), then we have \( \alpha = \beta + \gamma \) a contradiction to \( \alpha \in S_t \). If \( \gamma \in (-R^+_t) \), then \( -\gamma \in R^+_t \) and we have \( \beta = \alpha + (-\gamma) \) a contradiction to \( \beta \in S_t \). \( \square \)

Lemma 11.6.7 Let \( t \in V^\vee \) and \( A \subset V \) such that
- \( \langle t, \alpha \rangle > 0 \) for \( \alpha \in A \);
- \( (\alpha, \beta) \leq 0 \) for \( \alpha, \beta \in A \),
then the elements in \( A \) are linearly independent.

Proof. Let us consider a relation \( \sum_{\alpha \in A} a_\alpha \alpha = 0 \). Let \( B \) be the subset of elements \( \alpha \in A \) with \( a_\alpha \geq 0 \) and \( C \) be the subset of elements \( \alpha \in A \) with \( a_\alpha \leq 0 \). Define \( b_\alpha = a_\alpha \) for \( \alpha \in B \) and \( c_\alpha = -a_\alpha \) for \( \alpha \in C \), we have \( b_\alpha \) and \( c_\alpha \) non negative. We obtain a relation
\[
\sum_{\alpha \in B} b_\alpha \alpha = \sum_{\alpha \in C} c_\alpha \alpha.
\]
Lemma 11.6.10
Proof. Let \( \beta \) be a root and hence \( \langle \beta, \alpha \rangle \geq 0 \) for all \( \alpha \in R \), \( \beta \in R \). Therefore, \( \sum_{\alpha \in B} b_{\alpha} \alpha = 0 \). Applying \( \langle t, \alpha \rangle \), we get
\[
0 = \langle t, \sum_{\alpha \in B} b_{\alpha} \alpha \rangle = \sum_{\alpha \in B} b_{\alpha} \langle t, \alpha \rangle
\]
and because \( \langle t, \alpha \rangle > 0 \) for all \( \alpha \in A \) and \( b_{\alpha} \geq 0 \), we have \( b_{\alpha} = 0 \) for all \( \alpha \in B \). The symmetric argument gives that \( c_{\alpha} = 0 \) for all \( \alpha \in C \).
\[ \square \]

The above three lemmas prove that \( S_t \) is a base for \( R \). Let us now prove that any base \( S \) for \( R \) is of the form \( S_t \) for some \( t \in V^\vee \). Indeed, let \( t \in V^\vee \) such that \( \langle t, \alpha \rangle > 0 \) for all \( \alpha \in S \). This is possible because \( S \) is a base of \( V \). Let us denote by \( R^+ \) the set of all roots \( \alpha \in R \) such that \( \alpha \) can be written as a linear combination of elements in \( S \) with non negative coefficients. We have \( R^+ \subset R^+_t \) and \((-R^+) \subset (-R^+_t)\) therefore we have equalities in both inclusions. Let \( \alpha \in S \), then \( \alpha \) is indecomposable in \( R^+_t \). Indeed, if \( \alpha = \beta + \gamma \) with \( \beta, \gamma \in R^+_t \), then \( \beta \) and \( \gamma \) are linear combination of elements in \( S \) with non negative coefficients, therefore so is \( \alpha \). But the only combination for \( \alpha = \alpha \), therefore \( \{\beta, \gamma\} = \{\alpha, 0\} \) a contradiction. We thus have \( S \subset S_t \) but they have the same number of elements are they are basis for \( V \) and we have \( S = S_t \).
\[ \square \]

From now on we fix a base \( S \) of the root system \( R \).

Definition 11.6.8 We denote by \( R^+ \) the set of roots \( \alpha \in R \) which can be written as a linear combination of elements in \( S \) with non negative coefficients. The elements in \( R^+ \) are called positive roots. The elements in \( R^- = (-R^+) \) are called the negative roots.

Proposition 11.6.9 Any positive \( \alpha \) root can be written as \( \alpha = \sum_{i=1}^{k} \alpha_i \) with \( \alpha_i \in S \) such that for all \( j \in [1, k] \) the sum \( \sum_{i=1}^{j} \alpha_i \) is a root.

Proof. Let \( t \in V^\vee \) be the element defined by \( \langle t, \beta \rangle = 1 \) for all \( \beta \in S \). We proceed by induction on \( \langle t, \alpha \rangle \). For \( \langle t, \alpha \rangle = 1 \), then \( \alpha \in S \) and the result follows. Assume that the result holds for all root \( \beta \in R^+ \) such that \( \langle t, \beta \rangle < \langle \alpha, t \rangle \). We may assume that \( \alpha \notin S \).

Lemma 11.6.10 There exists a simple root \( \beta \) such that \( \langle \alpha, \beta \rangle > 0 \).

Proof. Otherwise, by Lemma 11.6.7, the set \( S \cup \{\alpha\} \) would be linearly independent, a contradiction to the fact that \( \alpha \notin S \) and that \( S \) is a base.
\[ \square \]

There is therefore a simple root \( \alpha_k \in S \) such that \( \langle \alpha, \alpha_k \rangle > 0 \). But this implies that \( \alpha = \alpha_k \) is a root and \( \langle t, \alpha - \alpha_k \rangle = \langle t, \alpha \rangle - 1 > 0 \) therefore \( \alpha - \alpha_k \in R^+ \) and we apply the induction hypothesis on \( \alpha - \alpha_k \) to conclude the proof.
\[ \square \]

Proposition 11.6.11 If \( R \) is reduced, then for \( \alpha \in S \), the symmetry \( s_\alpha \) associated with \( \alpha \) leaves \( R \setminus \{\alpha\} \) invariant.

Proof. Let \( \beta \in R \setminus \{\alpha\} \) and write \( \beta = \sum_{\gamma \in S} a_\gamma \gamma \) with \( a_\gamma \geq 0 \). There is a simple root \( \gamma_0 \) different from \( \alpha \) such that \( a_{\gamma_0} > 0 \). Now we have
\[
\begin{align*}
s_\alpha(\beta) &= \sum_{\gamma \in R} a_\gamma s_\alpha(\gamma) = \sum_{\gamma \in R} a_\gamma \gamma - \left( \sum_{\gamma \in R} a_\gamma \langle \alpha^\vee, \gamma \rangle \right) \alpha.
\end{align*}
\]
Therefore, the coefficient of $\gamma_0$ is again $a_{\gamma_0} > 0$ for the root $s_\alpha(\beta)$ which has to be positive and different from $\alpha$.

**Corollary 11.6.12** If $R$ is reduced and let $\rho$ be half the sum of the positive roots. We have the equality $s_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in S$.

**Proof.** We have by definition the following formula for $\rho$ and we define $\rho_\alpha$ by the right hand side formula:

$$\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta \quad \text{and} \quad \rho_\alpha = \frac{1}{2} \sum_{\beta \in R^+ \setminus \{\alpha\}} \beta.$$ 

Therefore $\rho = \rho_\alpha + \alpha/2$. By the above proposition, we have $s_\alpha(\rho_\alpha) = \rho_\alpha$, therefore we have the equality $s_\alpha(\rho) = \rho_\alpha - \alpha/2 = \rho - \alpha$.

**Proposition 11.6.13** If $R$ is reduced, then the set $S^\vee = \{\alpha^\vee / \alpha \in S\}$ is a root system for the dual root system $R^\vee$.

**Proof.** Recall that we fixed an invariant bilinear form $(\ , \ )$ on $V$ identifying $V$ with $V^\vee$ via the map $\Phi : V \to V^\vee$ defined by $u \mapsto (v \mapsto (u,v))$. We therefore also have an invariant bilinear form on $V^\vee$ defined by $((\varphi, \psi)) = (\Phi^{-1}(\varphi), \Phi^{-1}(\psi))$. Remark that under this identification, we have

**Fact 11.6.14**

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \Phi(\alpha).$$

**Proof.** Indeed, we have seen that the following formula $\langle \alpha^\vee, v \rangle = \frac{2}{(\alpha, \alpha)} (\alpha, v)$ holds for any $v \in V$, the result follows.

In particular, we know that the set of roots $S^\vee$ is a basis of the vector space $V^\vee$ because non-zero multiple of its elements are the image of the basis $S$ by $\Phi$. In symbols,

$$S^\vee = (\alpha^\vee)_{\alpha \in S} = \left( \frac{2}{(\alpha, \alpha)} \Phi(\alpha) \right)_{\alpha \in S}.$$ 

Furthermore, we know that $S$ is a basis for the roots system therefore any root $\beta \in R$ can be written as a sum

$$\beta = \sum_{\alpha \in S} a_\alpha \alpha$$

where the coefficients $a_\alpha$ are integers all non negative or non positive at the same time. We get

$$\beta^\vee = \frac{2}{(\beta, \beta)} \Phi(\beta) = \frac{2}{(\beta, \beta)} \Phi \left( \sum_{\alpha \in S} a_\alpha \alpha \right) = \frac{2}{(\beta, \beta)} \sum_{\alpha \in S} a_\alpha \Phi(\alpha) = \sum_{\alpha \in S} a_\alpha (\alpha, \alpha) \beta^\vee$$

therefore the coefficients are all non negative or non positive at the same time. We however *a priori* do not know if these coefficients are integers.

To prove this, fix $t \in V^\vee$ such that $\langle t, \alpha \rangle = 1$ for $\alpha \in S$. We have $S = \{ \alpha \in R / \langle t, \alpha \rangle = 1 \}$ and for $\beta \in R$, we have $\langle t, \beta \rangle \in \mathbb{Z}$. Let $T = \Phi^{-1}(t) \in V$. We have the equality

$$\langle t, \alpha \rangle = \langle T, \alpha \rangle = \frac{(\alpha, \alpha)}{2} \langle \alpha^\vee, T \rangle$$
therefore we have the equality \( (R^\vee)^+_T = \{ \alpha^\vee \in R^\vee / \alpha \in R^+_T \} \). The set \( S^\vee_T \) of indecomposable elements in \( (R^\vee)^+_T \) is a basis for \( R^\vee \). By definition, \( S^\vee \) is contained in \( (R^\vee)^+_T \). By what we proved above, the cone generated by \( S^\vee \) is equal to the cone generated by \( (R^\vee)^+_T \). Therefore, the extremal rays of these cones, which are the half line generated by the elements in \( S^\vee \) or in \( S^\vee_T \) are the same. The elements in \( S^\vee \) and in \( S^\vee_T \) are therefore proportional and because \( R \) is reduced, the result follows.

The next result describes how the Weyl group acts on the set of basis.

**Theorem 11.6.15** Assume that \( R \) is reduced, let \( W \) be its Weyl group and fix \( S \) a basis of \( R \).

(i) For any \( t \in V^\vee \), there exists an element \( w \in W \) such that \( \langle w(t), \alpha \rangle \geq 0 \) for all \( \alpha \in S \).

(ii) If \( S' \) is another basis of \( R \), there exists \( w \in W \) such that \( w(S') = S \).

(iii) For each \( \beta \in R \), there exists \( w \in W \) such that \( w(\beta) \in S \).

(iv) The group \( W \) is generated by the symmetries \( s_\alpha \) for \( \alpha \in S \).

**Proof.** Let us define \( W_S \) to be the subgroup of \( W \) generated by the symmetries \( s_\alpha \) for \( \alpha \in S \). We first prove (i), (ii) and (iii) and then prove that \( W = W_S \).

(i) For \( W_S \). We proceed by induction on the number \( n(t) \) of simple roots \( \alpha \) such that \( \langle t, \alpha \rangle < 0 \). If \( n(t) = 0 \), then just take \( w = \text{Id} \). Assume that the result is true for \( n(t) = n \) and let \( t \) such that \( n(t) = n + 1 \). Let \( \alpha \) be a simple root with \( \langle t, \alpha \rangle < 0 \). Then we consider \( s_\alpha(t) \). We have \( \langle s_\alpha(t), \alpha \rangle = \langle t, s_\alpha(\alpha) \rangle = -\langle t, \alpha \rangle > 0 \) and for \( \beta \in S \setminus \{ \alpha \} \), we have \( \langle s_\alpha(t), \beta \rangle = \langle t, s_\alpha(\beta) \rangle \). But on the set \( S \setminus \{ \alpha \} \), the linear form \( \langle t, \alpha \rangle \) takes \( n \) times a negative value and because \( s_\alpha(S \setminus \{ \alpha \}) = S \setminus \{ \alpha \} \) the same is true for \( s_\alpha(t) \). Therefore we have \( n(s_\alpha(t)) = n \) and the result follows by induction.

(ii) If \( S' \) is a basis, there exists an element \( t \in V^\vee \) such that \( S' = S_t \). Let \( w \in W_S \) given by (i) such that \( \langle w(t), \alpha \rangle \geq 0 \) for \( \alpha \in S \). The choice of \( t \) also implies that \( \langle t, \beta \rangle \neq 0 \) for \( \beta \in R \) therefore \( \langle w(t), \alpha \rangle > 0 \) for \( \alpha \in S \). The set of positive roots \( R^+ \) is \( R^+_{w(t)} \) therefore \( S = S_{w(t)} \). This gives \( S = S_{w(t)} = w(S(t)) = w(S') \).

(iii) Let \( t \) be a linear form such that \( \langle t, \alpha \rangle \neq 0 \) for all \( \alpha \in R \) and such that

\[ ||t, \beta|| = \min\{||t, \alpha|| / \alpha \in R\} \]

and \( \langle t, \beta \rangle > 0 \). We prove that \( \beta \in S_t \) and the result will follow from (ii). Assume we have \( \beta = \alpha + \gamma \) with \( \alpha \) and \( \gamma \) in \( R^+_T \). We have \( \langle t, \beta \rangle = \langle t, \alpha \rangle + \langle t, \gamma \rangle \) therefore \( 0 < \langle t, \alpha \rangle = \langle t, \beta \rangle - \langle t, \gamma \rangle < \langle t, \beta \rangle \) a contradiction to the minimality.

(iv) We need to prove that \( s_\beta \in W_S \) for any root. By (i), there exists \( w \in W_S \) such that \( w(\beta) = \alpha \in S \), therefore we have \( w(s_\beta w^{-1} = s_\alpha \in W_S \) and the result follows.

**Definition 11.6.16** The set \( C_S \) of elements \( t \) in \( V^\vee \) such that \( \langle t, \alpha \rangle > 0 \) for all \( \alpha \in S \) is called the Weyl chamber associated to \( S \).

**Remark 11.6.17** The Weyl chambers are the connected components of

\[ V^\vee \setminus \bigcup_{\alpha \in R} \alpha^\perp \]

where \( \alpha^\perp = \{ t \in V^\vee / \langle t, \alpha \rangle = 0 \} \). The previous result proves that the Weyl group \( W \) act transitively on the Weyl chambers.

We shall prove later (if time permits) the following:

**Theorem** 11.6.18 The group \( W \) acts simply transitively on the set of Weyl chambers.
Remark 11.6.19 One can be more precise and prove that the Weyl group is generated by the symmetries $s_{\alpha}$ with $\alpha \in S$ and that the only relations among the $(s_{\alpha})_{\alpha \in S}$ are the relations

$$(s_{\alpha}s_{\beta})^{m(\alpha,\beta)} = 1$$

with $m(\alpha,\beta) = 2, 3, 4$ or 6 if the angle between $\alpha$ and $\beta$ is $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$.

11.7 The Cartan matrix

Definition 11.7.1 The Cartan matrix for the root system associated to the base $S$ is by definition the matrix $(\langle \langle \beta', \alpha \rangle \rangle)_{\alpha,\beta \in S}$.

Example 11.7.2 Consider the rank two root systems as in Example 11.1.8 and choose a base $S$ as in Example 11.6.2. The Cartan matrix for type $A_2$, $B_2 = C_2$ or $G_2$ are the following matrices:

$$
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
2 & -2 \\
-1 & 2
\end{pmatrix} \quad \text{or} \quad
\begin{pmatrix}
2 & -3 \\
-1 & 2
\end{pmatrix}.
$$

Proposition 11.7.3 Let $R$ and $R'$ be two reduced root systems in two vector spaces $V$ and $V'$. Let $S$ and $S'$ be basis for $R$ and $R'$ and let $\phi : S \rightarrow S'$ be a bijection such that $\langle \langle \phi(\alpha)' , \phi(\beta) \rangle \rangle = \langle \langle \alpha' , \beta \rangle \rangle$. Then there exists a unique isomorphism $f : V \rightarrow V'$ such that $f|_S = \phi$ and realising a bijection from $R$ to $R'$.

Proof. Because $S$ and $S'$ are basis of $V$ and $V'$, there exists a unique isomorphism $f : V \rightarrow V'$ such that $f|_S = \phi$. Let us prove that $f$ maps bijectively $R$ into $R'$. For this we prove the comutation relation: $s_{\phi(\alpha)} \circ f = f \circ s_{\alpha}$. To prove this we only need to check on a basis, for example on $S$. We have for $\beta \in S$:

$$s_{\phi(\alpha)}(f(\beta)) = s_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle \phi(\alpha)' , \phi(\beta) \rangle \phi(\alpha) = f(\beta) - \langle \phi(\alpha)' , \beta \rangle f(\alpha) = f(s_{\alpha}(\beta))$$

and the commutation relation holds. Therefore if $W$ resp. $W'$ is the Weyl group of $R$ resp. $R'$, then we have $W' = fWf^{-1}$, therefore, because $W(S) = R$ and $W'(S') = R'$ we get $f(R) = R'$.  

Corollary 11.7.4 A reduced root system is determined by its Cartan matrix.

Definition 11.7.5 The subgroup $\text{Out}(R)$ of the group $\text{Aut}(S)$ of bijections of $S$ defined by

$$\text{Out}(R) = \{ \sigma \in \text{Aut}(S) \mid \langle \langle \sigma(\alpha)' , \sigma(\beta) \rangle \rangle = \langle \langle \alpha' , \beta \rangle \rangle \text{ for all } \alpha \text{ and } \beta \text{ in } S \}$$

is called the group of outer automorphism of the root system.

Corollary 11.7.6 The group $\text{Out}(R)$ is the subgroup of $\text{Aut}(R)$ leaving $S$ invariant.

Proposition* 11.7.7 The group $\text{Aut}(R)$ is the semidirect product of $\text{Out}(R)$ and $W$.

Proof. First remark that $W$ is normal in $\text{Aut}(R)$. Indeed, if $f \in \text{Aut}(R)$ and if $\alpha \in R$ is a root, then $f(s_{\alpha}f^{-1}) = s_{f(\alpha)}$. Let $w$ be in the intersection $W \cap \text{Out}(R)$, then $w$ maps the Weyl chamber defined by $C$ to itself therefore $w = 1$ by Theorem* 11.6.18. Furthermore, for $f \in \text{Aut}(R)$, we have that $f(S)$ is a base for $R$ therefore, there exists $w \in W$ such that $w(f(S)) = S$ thus $w \circ f \in \text{Out}(R)$ and the result follows.

Corollary* 11.7.8 The group $\text{Out}(R)$ is isomorphic to the quotient $\text{Aut}(R)/W$. 

11.8 The Coxeter graph

Definition 11.8.1 A Coxeter graph is a finite graph such that the vertices are linked by 0, 1, 2 or 3 edges.

Definition 11.8.2 Let \( R \) be a root system and let \( S \) be a base of \( R \). The Coxeter graph of \( R \) with respect to \( S \) is the graph whose vertices are the elements in \( S \) and such that two vertices \( \alpha \) and \( \beta \) in \( S \) are linked by \( \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \) vertices.

Remark 11.8.3 Recall that we have \( \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \) is an integer with value \( 4 \cos^2 \phi \) where \( \phi \) is the angle between \( \alpha \) and \( \beta \) so that, because \( \alpha \) and \( \beta \) are not collinear, the above definition defines a Coxeter graph.

Lemma 11.8.4 If \( S \) and \( S' \) are two basis of \( R \), then the Coxeter graph of \( R \) associated to \( S \) and \( S' \) are isomorphic.

Proof. We know that there is an element \( w \in W \) such that \( w(S) = S' \). This element \( w \) induces a bijection from \( S \) to \( S' \).

Fact 11.8.5 For \( w \) in \( W \), \( f \in V^\vee \) and \( v \in V \), we have \( \langle w(f), w(v) \rangle = \langle f, v \rangle \).

Proof. We only need to check that \( \langle s_\alpha(f), v \rangle = \langle f, s_\alpha(v) \rangle \) for any root \( \alpha \). But we have
\[
\langle s_\alpha(f), v \rangle = \langle f - \langle f, \alpha \rangle \alpha^\vee, v - \langle \alpha^\vee, \alpha \rangle \alpha \rangle \\
= \langle f, v \rangle - \langle f, \alpha \rangle \langle \alpha^\vee, v \rangle - \langle f, \alpha \rangle \langle \alpha^\vee, \alpha \rangle + \langle f, \alpha \rangle \langle \alpha^\vee, v \rangle \langle \alpha^\vee, \alpha \rangle \\
= \langle f, v \rangle - \langle f, \alpha \rangle \langle \alpha^\vee, v \rangle - \langle f, \alpha \rangle \langle \alpha^\vee, \alpha \rangle + 2 \langle f, \alpha \rangle \langle \alpha^\vee, v \rangle \\
= \langle f, v \rangle.
\]
The result follows.

Wew therefore have \( \langle w(\alpha^\vee), w(\beta) \rangle \langle w(\beta^\vee), w(\alpha) \rangle = \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \), giving that the two Coxeter graph are isomorphic.

Example 11.8.6 The Coxeter graphs of roots systems of type \( A_1, A_1 \times A_1, A_2, B_2 = C_2 \) and \( G_2 \) are the following:

\[
\begin{array}{cccccc}
& & & & & \\
& o & o & o & o & o \\
A_1 & A_1 \times A_1 & A_2 & B_2 = C_2 & G_2
\end{array}
\]

11.9 Irreducible root systems

Proposition 11.9.1 Fort \( i \in \{1, 2\} \), let \( R_i \) a root system in a vector space \( V_i \). Let \( V \) be the direct sum of \( V_1 \) and \( V_2 \) and identify \( R_i \) as subsets of \( V \). Then the union \( R = R_1 \cup R_2 \) is a root system in \( V \).

Proof. (1) We have that \( R \) is finite, spans \( V \) and does not contain 0.

(2) For \( \alpha \) an element in \( R \) then \( \alpha \) is in \( R_1 \) or in \( R_2 \). Say it is in \( R_1 \), then there is a symmetry \( s_\alpha \) on \( V_1 \) and we extend this symmetry on \( V \) by the identity on \( V_2 \). This defines a symmetry on \( V \) mapping \( R_1 \) to \( R_1 \) because \( s_\alpha \) does on \( V_1 \) and \( R_2 \) on \( R_2 \) because it is the identity on \( V_2 \).

(3) Let \( \alpha \) and \( \beta \) be elements in \( R \). If both are in \( R_1 \) or in \( R_2 \), then \( s_\alpha(\beta) - \beta \) is an integer multiple of \( \alpha \) because \( R_1 \) and \( R_2 \) are root systems. If \( \alpha \in R_1 \) and \( \beta \in R_2 \), then \( s_\alpha(\beta) = \beta \) and \( s_\alpha(\beta) - \beta = 0 \) is an integer multiple of \( \alpha \).
Definition 11.9.2 A root system $R$ in $V$ is called reducible if there exists a non trivial direct sum $V = V_1 \oplus V_2$ with $R_1 = V_1 \cap R$ and $R_2 = V_2 \cap R$ root systems in $V_1$ and $V_2$. A root system is called irreducible if it is not reducible.

Proposition 11.9.3 Let $R$ be a root system in $V$ and suppose that $V$ is a direct sum of $V_1$ and $V_2$ such that $R$ is contained in $V_1 \cup V_2$. Let $R_i = V_i \cap R$ for $i \in \{1, 2\}$.

(i) The spaces $V_1$ and $V_2$ are orthogonal for any invariant bilinear form.

(ii) For $i \in \{1, 2\}$, the subset $R_i$ is a root system in $V_i$.

Proof. (i) Let $\alpha \in R_1$ and $\beta \in R_2$. We have that $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$ is a root therefore in $V_1$ or in $V_2$. But $\beta$ in is $V_2$ and non zero therefore $s_\alpha(\beta) \in V_2$ which implies that $\langle \alpha^\vee, \beta \rangle \alpha \in V_1 \cap V_2 = 0$ thus $\langle \alpha^\vee, \beta \rangle = 0$ and for any invariant form we have $\langle \alpha, \beta \rangle = 0$. Remark that $R_1$ and $R_2$ span $V_1$ and $V_2$ (because $R$ spans $V$), the result follows.

(ii) We know that $R_i$ is finite and does not contain $0$ (because $R$ is finite and does not contain $0$) and because $R$ spans $V$, the $R_i$ respectively span $V_i$. Let $\alpha \in R_1$, then $\alpha$ is a root in $R$ therefore $s_\alpha$ is a bijection of $R_1 \cup R_2$. But because $\langle \alpha^\vee, \beta \rangle = 0$ for $\beta \in R_2$, we have that $s_\alpha$ is the identity on $R_2$ and therefore maps $R_1$ on itself. The same is true for $\alpha \in R_2$. Finally, if $\alpha$ and $\beta$ are in $R_1$, then $s_\alpha(\beta) - \beta$ is an integer multiple of $\alpha$ because this is true for $R$. \qed

Corollary 11.9.4 A root system $R$ in $V$ is reducible if and only if there exists a non trivial decomposition $V = V_1 \oplus V_2$ such that $R$ is contained in $V_1 \cup V_2$.

Proposition 11.9.5 A root system is irreducible if and only if its Coxeter graph is non empty and connected.

Proof. If $R$ is reducible, write $R = R_1 \cup R_2$ and we take $S = S_1 \cup S_2$ where $S_i$ is a basis of $R_i$ for $i \in \{1, 2\}$. This is easily seen to be a basis of $R$. By the above proposition, we see that the Coxeter graph is not connected: the graphs with vertices in $S_1$ and $S_2$ are disconnected.

Conversely, if the Coxeter graph of a root system $R$ in $V$ is not connected, then let $S = S_1 \cup S_2$ be a decomposition of a basis $S$ such that for $\alpha \in S_1$ and $\beta \in S_2$, we have $\langle \alpha, \beta \rangle = 0$. Let $V_1$ be the span of $S_1$ and let $V_2$ be the span of $S_2$. These spaces are non trivial and we need to prove that $R$ is contained in $V_1 \cup V_2$. But for $\alpha \in R$, there exists $w \in W$ such that $w(\alpha) \in S$, therefore $w(\alpha)$ is is $S_1$ or in $S_2$. Let us assume it is in $S_1$. We prove that $\alpha$ is in $V_1$. Indeed, the group $W$ is spanned by reflections $s_\beta$ for $\beta \in S$. We have $s_\beta(V_1) \subset V_1$ and $s_\beta(V_2) \subset V_2$ (because $s_\beta$ maps elements in $S_1$ resp. $S_2$ to linear combinaison of elements in $S_1$ resp. $S_2$). Therefore $\alpha = w^{-1}(w(\alpha))$ is in $V_1$. \qed