Chapter 1

First definitions and properties

1.1 Algebraic groups

1.1.1 Definitions

In this lectures, we will use basic notions of algebraic geometry. Our main reference for algebraic geometry will be the book [Har77] by R. Hartshorne. We will work over an algebraically closed field \( k \) of any characteristic. We will call variety a reduced separated scheme of finite type over \( k \).

The basic definition is the following.

**Definition 1.1.1** An algebraic group is a variety \( G \) which is also a group and such that the maps defining the group structure \( \mu : G \times G \to G \) with \( \mu(x, y) = xy \), the multiplication, \( i : G \to G \) with \( i(x) = x^{-1} \) the inverse and \( e_G : \text{Spec}(k) \to G \) with image the identity element \( e_G \) of \( G \) are morphisms.

There are several associated definitions.

**Definition 1.1.2**

(i) An algebraic group \( G \) is linear if \( G \) is an affine variety.

(ii) A connected algebraic group which is complete is called an abelian variety.

(iii) A morphism \( G \to G' \) of varieties between two algebraic groups which is a group homomorphism is called a homomorphism of algebraic groups.

(iv) A closed subgroup \( H \) of an algebraic group \( G \) is a closed subvariety of \( G \) which is a subgroup.

**Fact 1.1.3** Let \( G \) be an algebraic group and \( H \) a closed subgroup, then there is a unique algebraic group structure on \( H \) such that the inclusion map \( H \to G \) is a morphism of algebraic groups.

*Proof.* Exercise. \( \square \)

**Fact 1.1.4** Let \( G \) and \( G' \) be two algebraic groups. The product \( G \times G' \) with the direct product group structure is again an algebraic group. It is called the direct product of the algebraic groups \( G \) and \( G' \).

*Proof.* Exercise. \( \square \)

1.1.2 Chevalley’s Theorem

One usually splits the study of algebraic groups in two parts: the linear algebraic groups and the abelian varieties. This is because of the following result that we shall not try to prove.
Theorem 1.1.5 Let $G$ be an algebraic group, then there is a maximal linear algebraic subgroup $G_{\text{aff}}$ of $G$. This subgroup is normal and the quotient $A(G) := G/G_{\text{aff}}$ is an abelian variety. In symbols, we have an exact sequence of algebraic groups:

$$1 \to G_{\text{aff}} \to G \to A(G) \to 1.$$ 

Furthermore, the map $G \to A(G)$ is the Albanese map.

Let us now give the following result on abelian varieties.

Theorem 1.1.6 An abelian variety is a commutative algebraic group.

From now on we assume that all algebraic groups are affine.

1.1.3 Hopf algebras

Algebraic groups can be defined only by the existence of the morphisms $\mu : G \times G \to G$, $i : G \to G$ and $e_G : \text{Spec}(k) \to G$ such that the following diagrams are commutative. We denote by $\pi : G \to \text{Spec}(k)$ the structural map. In the last diagram, we identified $G$ with $G \times \text{Spec}(k)$ and $\text{Spec}(k) \times G$. If we assume that the algebraic group $G$ is linear, then $G = \text{Spec}(A)$ for some finitely generated algebra $A$ that we shall often denote by $k[G]$. The maps $\mu$, $i$, $e_G$ and $\pi$ define the following algebra morphisms: $\Delta : A \to A \otimes A$, $\iota : A \to A$ called the comultiplication, $\iota : A \to A$ called the antipode, $\epsilon : A \to k$ and $j : k \to A$. Let us furthermore denote by $m : A \otimes A \to A$ the multiplication in the algebra $A$ and recall that the corresponding morphism is the diagonal embedding $\text{Spec}(A) \to \text{Spec}(A) \times \text{Spec}(A)$. The above diagrams translate into the following commutative diagrams.

Definition 1.1.7 A $k$-algebra $A$ with morphisms $\Delta$, $\iota$, $\epsilon$, $j$ and $m$ as above is called a Hopf algebra.

Exercise 1.1.8 Give the meaning of a group morphism in terms of the map $\mu$, $i$ and $e_G$ and its interpretation in terms of Hopf algebras. This will be called a Hopf algebra morphism.

1.1.4 Examples

The first basic two examples are $G = \mathbb{A}^1 = k$ and $G = \mathbb{A}^1 \setminus \{0\} = k^\times$.

Example 1.1.9 In the first case we have $k[G] = k[T]$ for some variable $T$. The comultiplication is $\Delta : k[T] \to k[T] \otimes k[T]$ defined by $\Delta(T) = T \otimes 1 + 1 \otimes T$, the antipode $\iota : k[T] \to k[T]$ is defined by $\iota(T) = -T$ and the map $\epsilon : k[T] \to k$ is defined by $\epsilon(T) = 0$. This group is called the additive group and is denoted by $\mathbb{G}_a$.

Example 1.1.10 In the second case we have $k[G] = k[T, T^{-1}]$ for some variable $T$. The comultiplication is $\Delta : k[T, T^{-1}] \to k[T, T^{-1}] \otimes k[T, T^{-1}]$ defined by $\Delta(T) = T \otimes T$, the antipode $\iota : k[T, T^{-1}] \to k[T, T^{-1}]$ is defined by $\iota(T) = T^{-1}$ and the map $\epsilon : k[T, T^{-1}] \to k$ is defined by $\epsilon(T) = 1$. This group is called the additive group and is denoted by $\mathbb{G}_m$ or $\text{GL}_1$.

Example 1.1.11 For $n$ an integer, the $\mathbb{G}_m \to \mathbb{G}_m$ defined by $x \mapsto x^n$ is a group homomorphism. On the Hopf algebra level, it is given by $T \mapsto T^n$ if $k[\mathbb{G}_m] = k[T, T^{-1}]$.

Note that if $\text{char}(k) = p$ and $p$ divides $n$, then this morphism is bijective by is not an isomorphism.
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**Example 1.1.12** Consider the algebra $\mathfrak{gl}_n$ of $n \times n$ matrices and let $D$ be the polynomial computing the determinant of a matrix. The vector space $\mathfrak{gl}_n$ can be seen as an affine variety with $k[\mathfrak{gl}_n] = k[(T_{i,j})_{i,j \in [1,n]}]$. The general linear group $GL_n$ is the open set of $\mathfrak{gl}_n$ defined by the non vanishing of $\det = D(T_{i,j})$. We thus have $GL_n = \operatorname{Spec}(k[\mathfrak{gl}_n], \det^{-1})$.

The comultiplication $\Delta$ is given by

$$\Delta(T_{i,j}) = \sum_{k=1}^n T_{i,k} \otimes T_{k,j}.$$ 

The value of $\iota(T_{i,j})$ is the $(i, j)$-entry in the inverse matrix $(T_{k,l})^{-1}$ or of the matrix $\det^{-1} \operatorname{Com}(T_{k,l})$ where $\operatorname{Com}(M)$ is the comatrix of $M$. The map $\epsilon$ is given by $\epsilon(T_{i,j}) = \delta_{i,j}$.

Since $\mathfrak{gl}_n$ is irreducible of dimension $n^2$, so is $GL_n$.

**Exercise 1.1.13** Check that these maps indeed define the well known group structure on $GL_n$.

**Example 1.1.14** Any subgroup of $GL_n$ which is closed for the Zariski topology is again an algebraic group. For example:

- any finite subgroup;
- the group $D_n$ of diagonal matrices;
- the group $T_n$ of upper triangular matrices;
- the subgroup $U_n$ of $T_n$ of matrices with diagonal entries equal to 1;
- the special linear group $SL_n$ of matrices with determinant equal to 1;
- the orthogonal group $O_n = \{ M \in GL_n / ^t X X = 1 \}$;
- the special orthogonal group $SO_n = O_n \cap SL_n$;
- the symplectic group $Sp_{2n} = \{ X \in GL_{2n} / ^t X J X = J \}$ with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

For each simple Lie algebra, there exists at least one associated algebraic group. We shall see that conversely, any linear algebraic group is a closed subgroup of $GL_n$ for some $n$.

**Example 1.1.15** It is already more difficult to give the algebra of the group $PGL_n$ which is the quotient of $GL_n$ by its center $Z(GL_n) = \mathbb{G}_m$. One can prove for example that $PGL_n$ is the closed subgroup of $GL(\mathfrak{gl}_n)$ of algebra automorphisms of $\mathfrak{gl}_n$.

**Example 1.1.16** As last example, let us give a non linear algebraic group. If $X$ is an elliptic curve then it has a group structure and is therefore the first example of an abelian variety. The group structure is defined via the isomorphism $X \rightarrow \operatorname{Pic}^0(X)$ defined by $P \mapsto \mathcal{O}_X(P - P_0)$ where $P_0$ is a fixed point.
1.2 First properties

1.2.1 Connected components

**Proposition 1.2.1** Let $G$ be an algebraic group.

(i) There exists a unique irreducible component $G^0$ of $G$ containing the identity element $e_G$. It is a closed normal subgroup of $G$ of finite index.

(ii) The subgroup $G^0$ is the unique connected component containing $e_G$. The connected components and the irreducible components of $G$ coincide.

(iii) Any closed subgroup of $G$ of finite index contains $G^0$.

**Proof.** (i) Let $X$ and $Y$ be two irreducible components of $G$ containing $e_G$. The product $XY$ is the image of $X \times Y$ by $\mu$ and is therefore irreducible as well as its closure $\overline{XY}$. Furthermore $X$ and $Y$ are contained in $\overline{XY}$ (because $e_G$ is in $X$ and in $Y$). We thus have $X = \overline{XY} = Y$. This proves that there is a unique irreducible component $G^0 = X$ of $G$ containing $e_G$ and that it is stable under multiplication and closed. Therefore $G^0$ is a closed subgroup. Consider, for $g \in G$, the inner automorphism $\text{Int}(g) : G \to G$ defined by $x \mapsto gxg^{-1}$. We have that $\text{Int}(g)(G^0)$ is irreducible and contains $e_G$, therefore $\text{Int}(g)(G^0) \subset G^0$ and $G^0$ is normal.

Note that $G^0$ being irreducible, it is connected. Let $g \in G$, using the isomorphism $G \to G$ defined by $x \mapsto gx$, we see that the irreducible components of $G$ containing $g$ are in one-to-one correspondence with the irreducible components of $G$ containing $e_G$. There is a unique one which is $gG^0$. The irreducible components of $G$ are therefore the $G^0$ orbits and are thus disjoint. They must coincide with the connected components. Because there are finitely many irreducible components, the group $G^0$ must have finite index. This proves also (ii).

(iii) Let $H$ be a closed subgroup of finite index in $G$. Let $H^0$ be its intersection with $G^0$. The quotient $G^0/H^0$ is a subgroup of $G/H$ therefore finite. Thus $H^0$ is open and closed in $G^0$ thus $H^0 = G^0$ and the result follows. \qed

**Remark 1.2.2** Note that the former proposition imply that all the components of the group $G$ have the same dimension.

1.2.2 Image of a group homomorphism

**Lemma 1.2.3** Let $U$ and $V$ be dense open subsets of $G$, then $UV = G$.

**Proof.** Let $g \in G$, then $U$ and $gV^{-1}$ are dense open subset and must meet. Let $u$ be in the intersection, then there exists $v \in V$ with $u = gv^{-1} \in U$ thus $g = uv$. \qed

**Lemma 1.2.4** Let $H$ be a subgroup of $G$.

(i) The closure $\overline{H}$ of $H$ is a subgroup of $G$.

(ii) If $H$ contains a non-empty open subset of $\overline{H}$, then $H$ is closed.

**Proof.** (i) Let $h \in H$, then $hH \subset H \subset \overline{H}$ thus, because $h\overline{H}$ is the closure of $hH$ we have $h\overline{H} \subset \overline{H}$. This gives $h\overline{H} \subset \overline{H}$.

Now let $h \in \overline{H}$, by the last inclusion, we have $Hh \subset \overline{H}$ thus, because $\overline{H}h$ is the closure of $Hh$ we have $\overline{H}h \subset \overline{H}$. This gives $\overline{H}h \subset \overline{H}$.

Because $i$ is an isomorphism, we have $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$ proving the first part.

(ii) If $H$ contains a non-empty open subset $U$ of $\overline{H}$, then $H = \cup_{h \in H}hU$ is open in $\overline{H}$ and by the previous lemma, we have $\overline{H} = HH = H$. \qed
Proposition 1.2.5 Let $\phi : G \to G'$ be a morphism of algebraic groups.

(i) The kernel $\ker \phi$ is a closed normal subgroup.

(ii) The image $\phi(G)$ is a closed subgroup of $G$.

(iii) We have the equality $\phi(G^0) = \phi(G)^0$.

Proof. (i) The kernel is normal and the inverse image of the closed subset $\{e_{G'}\}$ therefore closed.

(ii) By Chevalley’s Theorem (in algebraic geometry, see [Har77, Exercise II.3.19]), the image $\phi(G)$ contains an open subset of its closure. By the previous lemma, it has to be closed.

(iii) $G^0$ being irreducible, the same is true for $\phi(G^0)$ which is therefore connected and thus contained in $\phi(G)^0$. Furthermore, we have that $\phi(G)/\phi(G^0)$ is a quotient of $G/G^0$ therefore finite. Thus $\phi(G^0)$ is of finite index in $\phi(G)$ and $\phi(G)^0 \subseteq \phi(G^0)$.

1.2.3 Subgroup generated by subvarieties

Proposition 1.2.6 Let $(X_i)_{i \in I}$ be a family of irreducible varieties together with morphisms $\phi_i : X_i \to G$. Let $H$ be the smallest closed subgroup containing the images $Y_i = \phi_i(X_i)$. Assume that $e_G \in Y_i$ for all $i \in I$.

(i) Then $H$ is connected.

(ii) There exist an integer $n$, a sequence $a = (a(1), \ldots, a(n)) \in \mathbb{N}$ and $\epsilon(k) = \pm 1$ for $k \in [1, n]$ such that $H = Y_{a(1)}^{\epsilon(1)} \cdots Y_{a(n)}^{\epsilon(n)}$.

Proof. Let us prove (ii), this will imply (i) since the $Y_i$ are irreducible.

Enlarging the family, we may assume that $Y_i^{-1} = Y_j$ for some $j$ and we get rid of the signs $\epsilon(k)$. For $a = (a(1), \ldots, a(n))$, let $Y_a = Y_{a(1)} \cdots Y_{a(n)}$. It is an irreducible variety as well as its closure $\overline{Y_a}$. Furthermore, we have by the same argument as is the former lemma the inclusion $\overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$. Let $a$ be such that $\overline{Y_a}$ is maximal for the inclusion i.e. for any $b$, we have $\overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_a}$. This is possible because the dimensions are finite. Now $\overline{Y_a}$ is irreducible, closed and closed under taking products. Note that for all $b$ we have $\overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_a}$ therefore because $e_G$ lies in all $Y_i$ we have $\overline{Y_b} \subseteq \overline{Y_a}$. Furthermore $\overline{Y_a}^{-1} = \overline{Y_a}$ and is the closure of the product $Y_{a(1)}^{-1} \cdots Y_{a(n)}^{-1}$ and thus contained in $\overline{Y_a}$. Therefore $\overline{Y_a}$ is a closed subgroup of $G$ containing the $Y_i$ thus $H \subseteq \overline{Y_a}$ but obviously $\overline{Y_a} \subset H$ so the result follows.

Corollary 1.2.7 (i) If $(G_i)_{i \in I}$ is a family of closed connected subgroups of $G$, then the subgroup $H$ generated by them is closed and connected. Furthermore, there is an integer $n$ such that $H = G_{a(1)} \cdots G_{a(n)}$.

Definition 1.2.8 Let $H$ and $K$ be subgroups of a group $G$, we denote by $(H, K)$ the subgroup generated by the elements $hkk^{-1}k^{-1}$ (called the commutators).

Corollary 1.2.9 If $H$ and $K$ are closed subgroups such that one of them is connected, then $(H, K)$ is closed and connected.

Proof. Assume that $H$ is connected. This follows from the previous proposition using the family $\phi_k : H \to G$ with $\phi_k(h) = hkk^{-1}k^{-1}$ which is indexed by $K$.
1.3 Action on a variety

1.3.1 Definition

Definition 1.3.1 (i) Let $X$ be a variety with an action of an algebraic group $G$. Let $a_X : G \times X \to X$ with $a_X(g, x) = g \cdot x$ be the map given by the action. We say that $X$ is a $G$-variety or a $G$-space if $a_X$ is a morphism.

(ii) A $G$-space with a transitive action of $G$ is called a homogeneous space.

(iii) A morphism $\phi : X \to Y$ between $G$-spaces is said to be equivariant if the following diagram commutes:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{a_X} & X \\
\downarrow{\text{Id} \times \phi} & & \downarrow{\phi} \\
G \times Y & \xrightarrow{a_Y} & Y
\end{array}
\]

(iv) Let $X$ be a $G$-space and $x \in X$. The orbit of $x$ is the image $G \cdot x = a_X(G \times \{x\})$. The isotropy group of $x$ or stabiliser of $x$ is the subgroup $G_x = \{g \in G / g \cdot x = x\}$.

Exercise 1.3.2 Prove that the stabiliser $G_x$ is the reduced scheme build on the fiber product $G_x = (G \times \{x\}) \times_X \{x\}$.

Example 1.3.3 The group $G$ can be seen as a $G$-space in several ways. Let $a_G : G \times G \to G$ be defined by $a_G(g, h) = ghg^{-1}$. The orbits are the conjugacy classes while the isotropy subgroups are the centralisers of elements.

Definition 1.3.4 If $X$ is a homogeneous space for the action of $G$ and furthermore all the isotropy subgroups are trivial, then we say that $X$ is a principal homogeneous space or torsor.

Example 1.3.5 The group $G$ can also act on itself by left (resp. right) translation i.e. $a_G : G \times G \to G$ defined by $a_G(g, h) = gh$ (resp. $a_G(g, h) = hg$). The action is then transitive and $G$ is a principal homogeneous space for this action.

Example 1.3.6 Let $V$ be a finite dimensional vector space then the map $a_V : \text{GL}(V) \times V \to V$ defined by $a_V(f, v) = f(v)$ defines a $\text{GL}(V)$-space structure on $V$.

Example 1.3.7 Let $V$ be a finite dimensional vector space and a homomorphism of algebraic group $r : G \to \text{GL}(V)$. Then the map $G \times V \to V$ given by the composition of $r \times \text{Id}$ with the map $a_V$ of the previous example defined a $G$-space structure on $V$. We also have a $G$-structure on $\mathbb{P}(V)$.

Definition 1.3.8 A morphism of algebraic groups $G \to \text{GL}(V)$ is called a rational representation of $G$ in $V$.

1.3.2 First properties

Lemma 1.3.9 Let $X$ be a $G$-space.

(i) Any orbit is open in its closure.

(ii) There is at least one closed orbit in $X$.

Proof. (i) An orbit $G \cdot x$ is the image of $G$ under the morphism $G \to X$ defined by $g \to g \cdot x$. By Chevalley’s theorem, we know that $G \cdot x$ contains an open subset $U$ of its closure. But then $G \cdot x = \bigcup_{g \in G} G \cdot x$ is open in $G \cdot x$. 

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(1) Let \( G \cdot x \) be an orbit of minimal dimension. It is open in \( G \cdot x \) therefore \( G \cdot x \setminus G \cdot x \) is closed of smaller dimension. However it is an union of orbits, therefore it is empty by minimality.

Let \( X \) be a \( G \)-space and assume that \( X \) is affine. Write \( X = \text{Spec} \, k[X] \). The action \( a_X : G \times X \to X \) is given by a map \( a_X^i : k[X] \to k[G] \otimes k[X] \). We may define a representation of abstract groups

\[
G \xrightarrow{r} \text{GL}(k[X])
\]

defined by \( (r(g)f)(x) = f(g^{-1}x) \). On the level of algebras, this map is defined as follows. An element \( g \in G \) defines a map \( \text{ev}_g : k[G] \to k \) and we can form the composition

\[
r(g) : k[X] \xrightarrow{a_X^i} k[G] \otimes k[X] \xrightarrow{\text{ev}_g^{-1}} k \otimes k[X] = k[X].
\]

**Proposition 1.3.10** Let \( V \) be a finite dimensional subspace of \( k[X] \).

1. There is a finite dimensional subspace \( W \) of \( k[X] \) which contains \( V \) and is stable under the action of \( r(g) \) for all \( g \in G \).
2. The subspace \( V \) is stable under \( r(g) \) for all \( g \in G \) if and only if we have \( a_X^i(V) \subset k[G] \otimes V \). In that case the map \( r_V : G \times V \to V \) defined by \( (g, f) \mapsto (\text{ev}_g \otimes \text{Id}) \circ a_X^i(f) \) is a rational representation.

**Proof.** (1) It is enough to prove this statement for \( V \) of dimension one. So let us assume that \( V \) is spanned by an element \( f \in k[X] \). Let us write

\[
a_X^i(f) = \sum_{i=1}^{n} v_i \otimes f_i
\]

with \( v_i \in k[G] \) and \( f_i \in k[X] \). For any \( g \in G \), we have

\[
r(g)f = \sum_{i=1}^{n} v_i(g)f_i
\]

therefore for all \( g \in G \), the element \( r(g)f \) is contained in the finite dimensional vector subspace of \( k[X] \) spanned by the elements \( (f_i)_{i \in [1,n]} \). Therefore the span \( W \) of the elements \( r(g)f \) for all \( g \in G \) is finite dimensional. This span is obviously spable under the action of \( r(g) \) for all \( g \in G \) since \( r(g)r(g')f = r(gg')f \).

(2) Assume that \( V \) is stable by \( r(g) \) for all \( g \in G \). Let us fix a base \( (f_i)_{i \in [1,n]} \) of \( V \) and complete it with the elements \( (g_j)_j \) to get a base of \( k[X] \). Let \( f \in V \) and write

\[
a_X^i(f) = \sum_{i=1}^{n} v_i \otimes f_i + \sum_{j} u_j \otimes g_j
\]

with \( v_i, u_j \in k[G] \). If for all \( g \in G \) we have \( r(g)f \in V \), then for all \( g \in G \), we have \( u_j(g^{-1}) = 0 \) thus \( u_j = 0 \) thus \( a_X^i(V) \subset k[G] \otimes V \).

Conversely, if \( a_X^i(V) \subset k[G] \otimes V \), then we may write

\[
a_X^i(f) = \sum_{i=1}^{n} v_i \otimes f_i
\]

with \( v_i \in k[G] \) and \( f_i \in V \). For any \( g \in G \), we have

\[
r(g)f = \sum_{i=1}^{n} v_i(g)f_i \in V
\]

and the result follows. \( \square \)
1.3.3 Affine algebraic groups are linear

In this section we consider the action of $G$ on itself by left and right multiplication. Let us fix some notation. We denote by $\lambda$ and $\rho$ the representations of $G$ in $\operatorname{GL}(k[G])$ induced by left and right action. That is to say, for $g \in G$, we define $\lambda(g) : k[G] \to k[G]$ and $\rho(g) : k[G] \to k[G]$. Explicitly, for $h \in G$ and for $f \in k[G]$, we have

$$(\lambda(g)f)(x) = f(g^{-1}x) \text{ and } (\rho(g)f)(x) = f(xg).$$

**Exercise 1.3.11** If $\iota : k[G] \to k[G]$ is the antipode isomorphism, then, for all $g \in G$, we have the equality $\rho(g) = \iota \circ \lambda(g) \circ \iota^{-1}$.

**Lemma 1.3.12** The representations $\lambda$ and $\rho$ are faithful.

*Proof.* We only deal with $\lambda$, the proof with $\rho$ is similar or we can use the former exercise. Let us assume that $\lambda(g) = e_{\operatorname{GL}(k[G])}$. Then $\lambda(g)f = f$ for all $f \in k[G]$. Therefore, for all $f \in k[G]$ we have $f(g^{-1}e_G) = f(e_G)$. This implies $g^{-1} = e_G$. \(\Box\)

**Theorem 1.3.13** Any linear algebraic group is a closed subgroup of $\operatorname{GL}_n$ for some $n$.

*Proof.* Let $V$ be a finite dimensional subspace of $k[G]$ which spans $k[G]$ as an algebra. By Proposition 1.3.10, there exists a finite dimensional subspace $W$ containing $V$ and stable under the action of $\lambda(g)$ for all $g \in G$. Let us choose a basis $(f_i)_{i \in [1,n]}$ of $W$. Because $W$ is stable, again by Proposition 1.3.10, we may write

$$a^g_W(f_i) = \sum_{j=1}^n m_{i,j} \otimes f_j$$

with $a^g_W : W \to k[G] \otimes W$ associated to the action $\lambda_W$ and $m_{i,j} \in k[G]$. We may define the following morphism

$$\phi^g : k[\operatorname{GL}_n] = k[[T_{i,j}],i,j \in [1,n], \det^{-1}] \to k[G]$$

by $T_{i,j} \mapsto m^{i,j}$ and $\det^{-1} \mapsto \det(m_{j,i})$ where here $m^{i,j}$ are the coefficients of the inverse of $(m_{i,j})$. On the level of points, this defines a morphism $\phi : G \to \operatorname{GL}_n$ given by $g \mapsto (m_{j,i}(g^{-1}))_{i,j \in [1,n]}$. Note that because $\lambda(gg')f = \lambda(g)\lambda(g')f$ we easily get that this map is a group morphism. We thus have a morphism of algebraic groups $\phi : G \to \operatorname{GL}_n$. Furthermore the image of $\phi^g$ contains the elements $f_i$ which generate $k[G]$ therefore $\phi^g$ is surjective and $\phi$ is an embedding. \(\Box\)

**Lemma 1.3.14** Let $H$ be a closed subgroup of $G$ and let $I_H$ be its ideal in $k[G]$. Then we have the equalities:

$$H = \{g \in G \mid \lambda(g)I_H = I_H\} = \{g \in G \mid \rho(g)I_H = I_H\}.$$

*Proof.* It is enough to prove it for $\lambda$. Let $g \in G$ with $\lambda(g)I_H = I_H$, then for all $f \in I_H$, we have $f(g^{-1}) = \lambda(g)f(e-G) = 0$ since $\lambda(g)f \in I_H$ and $e-G \in H$. Therefore $g^{-1} \in H$.

Conversely if $g \in H$, let $f \in I_H$ and $h \in H$. We have $\lambda(g)f(h) = f(g^{-1}h) = 0$ since $g^{-1}h \in H$. Therefore $\lambda(g)f \in I_H$. \(\Box\)