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On the quantum cohomology of adjoint varieties

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Abstract

We study the quantum cohomology of adjoint and coadjoint homogeneous spaces. The product of any two Schubert classes does not involve powers of the quantum parameter greater than 2. With the help of the quantum to classical principle, we give presentations of the quantum cohomology algebras. These algebras are semi-simple for adjoint non-coadjoint varieties and some properties of the induced strange duality are shown.

Contents

1. *Introduction*

In this paper, we study the quantum and classical cohomology rings of some rational homogeneous varieties that we call adjoint and coadjoint varieties (see Definition 2.1).

In recent works, Buch, Kresch and Tamvakis [**5**, **6**] studied the quantum cohomology of Grassmannians and isotropic Grassmannians, giving presentations of these rings and combinatorial rules for some Gromov–Witten invariants. These are homogeneous spaces under classical algebraic groups, and one can study their cohomology using the fact that they parametrize some linear spaces (for example, ker-span technique).

When one wants to study exceptional homogeneous spaces, similar methods are not available. Our strategy is to consider classes of homogeneous spaces, including some classical ones and some exceptional ones, and show that they share some geometric properties, so that their quantum cohomology rings also have some common features. Our work partially relies on the generalized Littlewood–Richardson rule proved in [**12**] giving a combinatorial formula for intersection numbers of three Λ-*(co)minuscule* (see Definition 2.10) Schubert classes. In the case of (co)minuscule homogeneous spaces that we studied in [**9**, **11**], all Schubert classes are Λ-(co)minuscule. In the case of adjoint and coadjoint spaces that we consider in this paper, all classes up to half the dimension are Λ-(co)minuscule (Corollary 2.12), allowing one to get a presentation of the classical cohomology ring.

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Moreover, as for the (co)minuscule case, we will explain here that the quantum Schubert calculus is simpler for these varieties, and give several techniques for computing Gromov–Witten invariants. First we use the quantum Chevalley formula given in [**16**] and the fact that the quantum cohomology has a basis indexed by some affine roots to get a very simple formula for the quantum multiplication by the hyperplane class in terms of these roots (see Theorem 3). We then prove a result on the powers of q, the quantum parameter, appearing in a product of two Schubert classes. For $X = G/P$ an adjoint or coadjoint variety (see Definition 2.1), we have the following vanishing result.

THEOREM 1. *Let X be an adjoint or a coadjoint homogeneous space and let σ_u and σ_v be Schubert classes in $QH^*(X, \mathbb{Z})$. Then if we write*

$$\sigma_u * \sigma_v = \sum_{d,w} c_{u,v}^w(d) q^d \sigma_w,$$

we have $c_{u,v}^w(d) = 0$ unless $d \leqslant 2$.

This theorem is the consequence of the geometric fact that, given three points in X, if there passes a curve of degree 3 through these points, then there are infinitely many such curves. In the adjoint case, this follows from a uniform decomposition of the Lie algebra of G under the action of a Levi factor of P. In the coadjoint case, this follows from an explicit description of the locus covered by the curves of degree 3 through three fixed points, using the fact that these varieties are generic hyperplane sections of Scorza varieties (see Definition 3.13).

Our last tool is the quantum to classical principle, as initiated in [**5**], see Corollary 4.3. With all these techniques, we are able to give presentations of the quantum cohomology rings of all adjoint and coadjoint homogeneous spaces. These results are presented in Section 5. We do not give here a presentation in the classical cases as they were obtained by Buch, Kresch and Tamvakis in [**6**] but focus on exceptional groups. Some computations were made using a program written in Java (available at www.math.sciences.univ-nantes.fr/˜chaput/quiver-demo.html), which may be of some interest for those who want to make explicit computations in these quantum cohomology rings.

Finally we focus on strange duality, a known phenomenon for (co)minuscule homogeneous spaces X: in that case, there exists an involution ι of the quantum cohomology algebra $QH^*(X, \mathbb{C})_{\text{loc}}$ with the quantum parameter q inverted sending classes of degree d to classes of degree $-d$. The existence of such an involution is a consequence of the semi-simplicity of the ring $QH^*(X, \mathbb{Z})_{\text{loc}}$. We conjectured in [**11**] that the only homogeneous spaces X with $QH^*(X, \mathbb{Z})_{\text{loc}}$ semi-simple are the (co)minuscule homogeneous spaces.

In Section 6 we are able to prove that, for many rational homogeneous spaces X with Picard number 1, the algebra $QH^*(X, \mathbb{Z})_{\text{loc}}$ is not semi-simple (see Theorem 5). However, we give counterexamples to our conjecture because for X an adjoint non-coadjoint variety, the localized algebra $QH^*(X, \mathbb{C})_{\text{loc}}$ is semi-simple (see Theorem 6). This implies the existence of an algebra involution ι as above. We then describe some properties of this involution. In particular, once specialized at $q = 1$, it is the identity on classes of degree multiple of $c_1(X)$ (Proposition 6.5). However, it is more complicated in general than the involution in the minuscule and cominuscule cases. In fact, in the (co)minuscule cases, ι maps a Schubert class to a multiple of a Schubert class. Here, we have the following theorem.

THEOREM 2. *For X an adjoint non-coadjoint rational homogeneous space, there exists a Schubert class σ such that $\iota(\sigma)$ is not a multiple of a Schubert class.*

2. *Minuscule, cominuscule, adjoint and coadjoint varieties*

Let G be a semi-simple algebraic group with root system R. Let T be a maximal torus in G and B be a Borel subgroup containing T. We denote by W the associated Weyl group, by Δ the root system and by S the set of simple roots. We denote by Δ_s and Δ_l the subsets of short and long roots, respectively. We refer to [**3**], for example, for more details on root systems. There is a natural involution, called the *Weyl involution* and denoted by ι, on the set of simple roots S defined by $\iota(\alpha) = -w_0(\alpha)$, where w_0 is the longest element of the Weyl group W. We shall denote by Θ the highest root and by θ the highest short root. For simply laced groups, we have $\Theta = \theta$. We denote by $\alpha_{\rm ad}$ any simple root such that $\langle \alpha_{\rm ad}, \Theta^\vee \rangle = 1$ (such a root is unique except in type A). Here we use the notation $\alpha^\vee$ for the coroot associated to α (see [**3**] for more details) If ϖ is a weight in R, then we denote by ϖ^* the coweight defined by the equality $\langle \alpha, \varpi^* \rangle = \langle \varpi, \alpha^\vee \rangle$ for each simple root α (thus ϖ^* depends on the choice of the simple roots).

2.1. *First definitions*

Definition 2.1. Let ϖ be a dominant weight, we shall call ϖ
- (i) *minuscule* if, for any positive root $\alpha \in R_+$, we have $\langle \varpi, \alpha^\vee \rangle \leqslant 1$;
- (ii) *cominuscule* if ϖ^* is minuscule for the dual root system;
- (iii) *quasi-minuscule* if, for any positive root $\alpha \in R_+$, we have $\langle \varpi, \alpha^\vee \rangle \leqslant 2$ with equality if and only if $\alpha = \varpi$;
- (iv) *adjoint* if it is equal to the highest root;
- (v) *coadjoint* if it is equal to the highest short root.

Remark 2.2. We have $\theta^* = \theta^\vee$ and this is the highest coroot. This explains why θ is called a coadjoint weight. A weight ϖ is quasi-minuscule if and only if it is minuscule or coadjoint. In simply laced cases, minuscule and cominuscule weights, adjoint and coadjoint weights coincide (Table 1).

Definition 2.3. Let P be the standard parabolic subgroup associated to a weight ϖ. We say that the homogeneous variety G/P is *minuscule*, *cominuscule*, *quasi-minuscule*, *adjoint* or *coadjoint* if the weight ϖ is minuscule, cominuscule, quasi-minuscule, adjoint or coadjoint, respectively.

Remark 2.4. (i) The adjoint homogeneous spaces correspond to the classical terminology of adjoint varieties also called projectivized minimal nilpotent orbits.

Table 1. *Minuscule, quasi-minuscule, cominuscule, adjoint and coadjoint weights.*

Type of G	Quasi-minuscule: Minuscule	Quasi-minuscule: Coadjoint	Cominuscule	Adjoint
A_n	$\varpi_1, \ldots, \varpi_n$	$\varpi_1 + \varpi_n$	$\varpi_1, \ldots, \varpi_n$	$\varpi_1 + \varpi_n$
B_n	ϖ_n	ϖ_1	ϖ_1	ϖ_2
C_n	ϖ_1	ϖ_2	ϖ_n	$2\varpi_1$
D_n	$\varpi_1, \varpi_{n-1}, \varpi_n$	ϖ_2	$\varpi_1, \varpi_{n-1}, \varpi_n$	ϖ_2
E_6	ϖ_1, ϖ_6	ϖ_2	ϖ_1, ϖ_6	ϖ_2
E_7	ϖ_7	ϖ_1	ϖ_7	ϖ_1
E_8	None	ϖ_8	None	ϖ_8
F_4	None	ϖ_4	None	ϖ_1
G_2	None	ϖ_1	None	ϖ_2

TABLE 2. *Invariants of adjoint and coadjoint varieties.*

	Adjoint				Coadjoint			
G	Θ	G/P_Θ	$\dim(G/P_\Theta)$	$c_1(G/P_\Theta,\Theta)$	θ	G/P_θ	$\dim(G/P_\theta)$	$c_1(G/P_\theta,\theta)$
A_n	$\varpi_1+\varpi_n$	$\mathbb{F}(1,n;n+1)$	$2n-1$	n				
B_n	ϖ_2	$\mathbb{G}_Q(2,2n+1)$	$4n-5$	$2n-2$	ϖ_1	$\mathbb{Q}^{2n-1}$	$2n-1$	$2n-1$
C_n	$2\varpi_1$	$\mathbb{P}^{2n-1}$	$2n-1$	n	ϖ_2	$\mathbb{G}_\omega(2,2n)$	$4n-5$	$2n-1$
D_n	ϖ_2	$\mathbb{G}_Q(2,2n)$	$4n-7$	$2n-3$				
E_6	ϖ_2		21	11				
E_7	ϖ_1		33	17				
E_8	ϖ_8		57	29				
F_4	ϖ_1		15	8	ϖ_4		15	11
G_2	ϖ_2		5	3	ϖ_1	$\mathbb{Q}^5$	5	5

(ii) The geometry of minuscule homogeneous varieties is better understood than the geometry of general homogeneous spaces (see, for example, [**29**, **31**] or [**27**]). Quasi-minuscule weights were in particular introduced to generalize the classical theory of standard monomials in minuscule varieties (see [**29**]) to other homogeneous spaces (see [**21**]).

REMARK 2.5. For minuscule and cominuscule homogeneous spaces, the quantum cohomology was described in [**5**] for classical groups and in [**9**] in general. So we mainly focus in what follows on adjoint and coadjoint homogeneous spaces except in Subsection 2.4 where we include the minuscule and cominuscule cases in our simplified version of the quantum Chevalley formula.

The adjoint and coadjoint varieties are tabulated in the following array (for the simply laced Lie algebras the notions of adjoint and coadjoint varieties coincide and therefore we leave the corresponding box empty) (Table 2).

In this array, we denote by $\mathbb{F}(1,n,n+1)$, $\mathbb{G}_Q(2,n)$, $\mathbb{G}_\omega(2,2n)$ and $\mathbb{Q}^n$ the point-hyperplane incidence in $\mathbb{P}^n$, the Grassmannian of 2-dimensional isotropic subspaces in an n-dimensional vector space endowed with a non-degenerate quadratic form, the Grassmannian of 2-dimensional isotropic subspaces in a $2n$-dimensional vector space endowed with a symplectic form and an n-dimensional quadric, respectively. The notation $c_1(X,\varpi)$ for ϖ a dominant weight and $X=G/P_\varpi$, where P_ϖ corresponds to ϖ, stands for the number, if it exists, such that $-K_X=c_1(X,\varpi)\cdot\mathcal{L}$, where $\mathcal{L}$ is the ample line bundle on X corresponding to ϖ. In Table 2, it is equal to the classical index of the Fano variety X in most cases but differs in a few cases. Note that, for adjoint varieties G/P_Θ, we always have the equality $\dim(G/P_\Theta)=2c_1(G/P_\Theta,\Theta)-1$ (this well-known fact can be seen as a consequence of Propositions 3.10 and 3.7). Note that, for adjoint and coadjoint varieties $X=G/P$, there exists a non-negative integer r such that $\dim(X)=2r+1$.

We now give a uniform formula for $c_1(G/P_\varpi,\varpi)$ for minuscule, cominuscule, adjoint, coadjoint weights ϖ.

LEMMA 2.6. *The following formulas yield the first Chern class of minuscule, cominuscule, coadjoint or adjoint varieties*:

	Minuscule	*Coadjoint*	*Cominuscule*	*Adjoint*
$c_1(X,\varpi)$	$\langle\rho,\theta^\vee\rangle+1$	$\langle\rho,\theta^\vee\rangle$	$\langle\rho,\Theta^\vee\rangle+1$	$\langle\rho,\Theta^\vee\rangle$
$c_1(X,\varpi)$	$\langle\Theta,\rho^*\rangle+1$	$\langle\Theta,\rho^*\rangle$		

Proof. The lemma follows from a case-by-case inspection using the values of $c_1(X)$ given above and the values for the (co)minuscule cases given in [**9**].

In all cases but the (co)adjoint case in type A and the adjoint case in type C, we have $c_1(X,\varpi) = c_1(X)$ and one can also argue as follows: first recall (see [**4**, p. 85]) that $c_1(X) = \langle \delta_P, \beta^\vee \rangle$, where $\delta_P = \rho + w^P(\rho)$ and w^P is the longest element in the Weyl group W_P of P, and $\beta^\vee$ is the only simple coroot such that $\langle \varpi, \beta^\vee \rangle = 1$ (in the (co)adjoint case in type A there are two such coroots and in the adjoint case in type C there is no such coroot). Thus $c_1(X)$ is equal to $1 + \langle \rho, w^P(\beta^\vee) \rangle$. Then it is easy to see that $\alpha^\vee = w^P(\beta^\vee)$ is the highest coroot with the following properties:

(i) $\langle \varpi, \alpha^\vee \rangle = 1$;

(ii) $|\alpha^\vee| = |\beta^\vee|$.

Let $\alpha^\vee_{\mathrm{ad}}$ be the only coroot such that $\langle \alpha_{\mathrm{ad}}, \Theta^\vee \rangle = 1$. To prove the first line of the lemma, it is enough to check that the coroot $\alpha^\vee$ is $\theta^\vee, \theta^\vee - \alpha^\vee_{\mathrm{ad}}, \Theta^\vee, \Theta^\vee - \alpha^\vee_{\mathrm{ad}}$, according to the four cases. For example, in the minuscule case, the simple root β with $\langle \varpi, \beta^\vee \rangle = 1$ is short and therefore $\beta^\vee$ is long; moreover, by definition of a minuscule weight we have $\langle \varpi, \gamma^\vee \rangle \leqslant 1$ for any coroot $\gamma^\vee$. It thus follows that $\alpha^\vee$ is the highest coroot. The other cases are similar.

To prove the second line, we may observe that the sum of the coefficients of the highest root is the same in a root system and its dual. □

2.2. *Roots and Schubert varieties in the adjoint and coadjoint cases*

In this subsection, we recall some results on Schubert varieties specific to the adjoint and coadjoint homogeneous spaces. We will use the (untwisted) affine root system associated to the root system Δ. We refer to [**18**] for details on affine root systems. We shall need the following.

NOTATION 2.7. (i) We define the affine weight ρ by $\langle \rho, \alpha^\vee \rangle = 1$ and the affine coweight $\rho^\vee$ by $\langle \alpha, \rho^\vee \rangle = 1$ for all simple root α of the affine Weyl group.

(ii) Recall that there is a natural order on the roots defined by $\alpha \leqslant \beta$ if $\beta - \alpha$ is in the monoid generated by positive roots.

(iii) For P a parabolic subgroup of G, we denote by W_P the Weyl group of P. It is a subgroup of W and we denote by W^P the subset of minimal length representatives of the quotient W/W_P in W.

FACT 2.8. (i) *Fix B a Borel subgroup of G contained in P. The Bruhat decomposition gives a cellular decomposition of G/P as follows*:

$$G/P = \coprod_{u \in W^P} BuP/P.$$

The B-orbits BuP/P are called the Schubert cells and their closures are the Schubert varieties. The classes of Schubert varieties, the Schubert classes, form a basis $(\sigma_u)_{u\in W^P}$ of $H^*(G/P,\mathbb{Z})$.

(ii) *Assume $X = G/P$ is a coadjoint (respectively, an adjoint) homogeneous space. Then, the map $W^P \to \Delta_s$ (respectively, $W^P \to \Delta_l$) defined by $u \mapsto \alpha = u(\varpi)$ is bijective.*

Proof. The first assertion is the Bruhat decomposition and can be found in [**3**]. The second assertion is a reformulation of the fact that the orbit of the point ϖ in X under the action of the Weyl group is the set of T-fixed points in X. □

Let $u \in W^P$ and $\alpha = u(\varpi)$. We denote by $X(\alpha)$ the associated Schubert variety and we set $\sigma_\alpha = \sigma_u$.

PROPOSITION 2.9. *Let $X = G/P$ be a coadjoint or an adjoint homogeneous space and let α be a root. Assume that α is short if X is coadjoint and long if X is adjoint. Denote by $X(\alpha)$ the associated Schubert variety. Then, the following statements hold.*

(i) *The integer* $\dim(X)$ *is odd, we define r by the equality* $\dim(X) = 2r + 1$.
(ii) *There is an equivalence*

$$X(\alpha) \subset X(\beta) \iff \begin{cases} \alpha \leqslant \beta & \text{for } \alpha \text{ and } \beta \text{ of the same sign,} \\ \mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta) \text{ is connected} & \text{for } \alpha \text{ negative and } \beta \text{ positive.} \end{cases}$$

(iii) *Denote by $X(\alpha)^\vee$ the Poincaré dual of $X(\alpha)$ and by i the Weyl involution, then we have the equality $X(\alpha)^\vee = X(-i(\alpha))$.*

Proof. All these results can be easily deduced from results in [**21**, Section 3.1] in the coadjoint case. Indeed, (i) follows from [**21**, Theorem 3.1], (ii) follows from the conjunction of [**21**, Theorems 3.1 and 3.10]. For (iii), recall that Poincaré duality is given on W^P by $u \mapsto w_0 u w_0 w_X$, where w_0 in the longest element in W and w_X the longest element in W^P. In terms of roots, if $\alpha = u(-\varpi)$, then the root corresponding to $X(\alpha)^\vee$ is

$$w_0 u w_0 w_X(-\varpi) = w_0 u w_0(\varpi) = w_0 u(-\varpi) = w_0(\alpha) = -i(\alpha).$$

To prove the corresponding statements in the adjoint case, little more is needed. Indeed, all the results only involve Weyl groups and are therefore true simply because the Weyl group of a root system and the Weyl group of its dual are isomorphic. □

2.3. *Description of ϖ-minuscule elements in the adjoint and coadjoint cases*

Let us recall, according to Dale Peterson [**28**, p. 273] (see also [**12**]), the following definition.

DEFINITION 2.10. Let $\Lambda = \sum_i \Lambda_i \varpi_i$ be a dominant weight (where $(\varpi_i)_i$ are the fundamental weights).

(i) An element $w \in W$ is Λ-minuscule if there exists a reduced decomposition $w = s_{i_1} \dots s_{i_l}$ such that, for any $k \in [1, l]$, we have $s_{i_k} s_{i_{k+1}} \dots s_{i_l}(\Lambda) = s_{i_{k+1}} \dots s_{i_l}(\Lambda) - \alpha_{i_k}$.
(ii) The element w is Λ-cominuscule if w is $(\sum \Lambda_i \varpi_i^\vee)$-minuscule.

PROPOSITION 2.11. *If X is coadjoint or adjoint, then any element w in W^P with $l(w) \leqslant r$ is ϖ-minuscule or ϖ-cominuscule, respectively.*

Proof. The proof of [**21**, Theorem 3.1] proves that, in the coadjoint case, such a u is ϖ-minuscule and that it is maximal for this property. The adjoint case is obtained from the previous one by taking the dual root system. □

We refer to [**12**] for a definition of jeu de taquin. As a consequence of the main result in [**12**], we have the following corollary.

COROLLARY 2.12. *Let G/P be a (co)adjoint variety of dimension $2r + 1$. Let u, v and w be elements in W^P; then the Littlewood–Richardson coefficient $c_{u,v}^w$ can be computed using jeu de taquin for $l(w) \leqslant r$.*

2.4. *Quantum Chevalley formula*

In this subsection, we shall give a simplified statement for the quantum Chevalley formula as formulated in [**16**]. We shall also see that our parametrization of Schubert classes by roots as in the previous subsection extends to the quantum monomials, which are by definition of the form $q^d\sigma_u$ for some integer d and some $u \in W^P$: these monomials are parametrized by the roots of the affine root system $\widehat{R}$ associated to R and this parametrization is compatible with the quantum Chevalley formula.

For the moment we only assume that X has Picard number 1. Let us first define an injection of the set of quantum monomials in the set of affine weights $P^{\rm aff}$.

DEFINITION 2.13. (i) We denote by $\mathcal{M}$ the set of quantum monomials $\sigma_w \cdot q^d$ (with q inverted). It is indexed by the set of pairs (w,d) in $W^P \times \mathbb{Z}$.

(ii) Let $w \in W^P$ and $d \in \mathbb{Z}$. To the quantum monomial $\sigma_w \cdot q^d$, we associate the affine weight $\eta(w,d) = u(\varpi) - d\delta$, where δ is the minimal positive imaginary root in $\widehat{R}$ (see, for example, [**18**]).

(iii) For $\pi = \eta(w,d)$ we define by $\sigma_\pi = \sigma_w \cdot q^d$ the corresponding quantum monomial.

(iv) For $\pi \in \eta(\mathcal{M})$ we denote by $l(\pi)$ the integer $l(w) + dc_1$, if (w,d) is the element such that $\eta(w,d) = \pi$.

REMARK 2.14. In the coadjoint case, $\eta(\mathcal{M})$ is the set of all short real affine roots while in the adjoint case it is the set of all long real affine roots.

PROPOSITION 2.15. *Let $\pi \in \eta(\mathcal{M})$ and let h be the hyperplane class; we have*

$$h * \sigma_\pi = \sum_{\gamma>0,\, l(s_\gamma(\pi))=l(\pi)+1} \langle \pi, \gamma^\vee \rangle \sigma_{s_\gamma(\pi)}.$$

Proof. This is the particular case, when the Picard number of X is 1, of [**15**, Theorem 10.1]. In fact, let us assume $\pi = \eta(w,0)$ with $w \in W^P$; according to this result, the quantum product of h and σ_w is

$$\sum \langle \varpi, \alpha^\vee \rangle \sigma_{us_\alpha} + \sum q^{d(\alpha)} \langle \varpi, \alpha^\vee \rangle \sigma_{us_\alpha}, \tag{1}$$

where α is a positive root, $d(\alpha) = \langle \varpi, \alpha^\vee \rangle$, and in the first and second summands we have $l(ws_\alpha) = l(w) + 1$ and $l(ws_\alpha) = l(u) + 1 - c_1 d(\alpha)$, respectively. We denote by A the set of these roots, and by A_1 and A_2 the set of roots in the first and second cases, respectively. We consider an injection $\varphi : A \to \widehat{R}$ by mapping $\alpha \in A_1$ to $w(\alpha)$ and $\alpha \in A_2$ to $w(\alpha) + \delta$.

Let $\alpha \in A$ and let us define $\gamma = \varphi(\alpha)$. We claim that, for $\alpha \in A_1$ and $\alpha \in A_2$, $s_\gamma w(\varpi)$ is equal to $ws_\alpha(\varpi)$ and $ws_\alpha(\varpi) - d(\alpha)\delta$, respectively. In fact the first case follows from the equality $ws_\alpha = ws_\alpha w^{-1} w = s_{w(\alpha)} w$. The second case may be computed as follows:

$$\begin{aligned} s_\gamma w(\varpi) &= s_{w(\alpha)+\delta}(\varpi) = w(\varpi) - \langle w(\alpha)^\vee, w(\varpi) \rangle (w(\alpha) + \delta) \\ &= w(\varpi) - \langle \alpha^\vee, \varpi \rangle w(\alpha) - d(\alpha)\delta = ws_\alpha(\varpi) - d(\alpha)\delta, \end{aligned}$$

where the last equality follows from the first case. Thus, in particular, we get $s_\gamma w(\varpi) = \eta(ws_\alpha, d(\alpha))$.

Moreover, it follows that γ is a positive root. In fact, in the second case it has positive coefficient on δ, and in the first case we have $s_\gamma w(\varpi) = w(\varpi) - d(\alpha)\gamma$ and $s_\gamma w(\varpi) = ws_\alpha(\pi) < w(\alpha)$, so that $\gamma > 0$.

Finally φ is a bijection from A to the set of positive roots γ such that $l(s_\gamma w(\pi)) = l(w(\pi)) + 1$, and the proposition is proved. □

Our next goal is to restrict the set of roots γ in the former proposition in the four cases that we consider.

LEMMA 2.16. *If $[a/b]$ denotes the integral part of a/b, then we have the following equalities*:

	Minuscule	*Coadjoint*
$\langle \varpi - \pi, \rho^\vee \rangle$	$l(\pi)$	$l(\pi) + [l(\pi)/c_1]$
	Cominuscule	*Adjoint*
$\langle \rho, (\varpi - \pi)^\vee \rangle$	$l(\pi)$	$l(\pi) + [l(\pi)/c_1]$

Proof. Note that we have $\langle \delta, \rho^\vee \rangle = \langle \Theta, \rho^\vee \rangle + 1$ since $\delta = \Theta + \alpha_0$. Thus, using Lemma 2.6 and the definition of $l(\pi)$ given in Definition 2.13, it is enough to prove the lemma when $\pi = \eta(w, 0)$.

In this case, assuming that we are in the minuscule case, if we write $w = s_{i_l} \dots s_{i_1}$, then we have $w(\varpi) = \varpi - \sum \alpha_{i_k}$, and so $\langle \varpi - w(\varpi), \rho^\vee \rangle = l = l(w)$. Thus the lemma is proved in this case. The cominuscule case follows by duality.

In the coadjoint case, if $l(w) < c_1$, then any reduced expression of w will have the same property as in the minuscule case and the same argument gives $\langle \varpi - w(\varpi), \rho^\vee \rangle = l(w)$. If $l(w) \geqslant c_1$, then we can find a reduced expression $w = s_{i_l} \dots s_{i_1}$ such that $w(\varpi) = \varpi - \sum d_k \alpha_{i_k}$, where $d_k = 1$ for all k but $k = c_1$, and $d_{c_1} = 2$. Thus $\langle \varpi - w(\varpi), \rho^\vee \rangle = l(w) + 1$ and the lemma is again true. The adjoint case follows by duality. □

DEFINITION 2.17. Let $\pi \in \eta(\mathcal{M})$ and γ be a root. We say that γ interacts with π if $\langle \pi, \gamma^\vee \rangle > 0$ and either:

(i) We are in the minuscule or cominuscule cases and γ is a simple root.
(ii) We are in the coadjoint or adjoint cases, c_1 does not divide $l(\pi) + 1$ and γ is a simple root.
(iii) We are in the coadjoint or adjoint cases and c_1 divides $l(\pi) + 1$; this implies that there exist a simple root α and an integer d such that $\pi = \alpha + d\delta$. In this case γ interacts with π if either $\gamma = \alpha$ or $\gamma = \alpha + \beta$, where β is another simple root.

We are now in a position to give a more explicit formula for the quantum Chevalley formula of Proposition 2.15.

THEOREM 3. *We have a formula*

$$h * \sigma_\pi = \sum_\gamma \langle \pi, \gamma^\vee \rangle \sigma_{s_\gamma(\pi)},$$

where the sum runs over all positive roots γ which interact with π.

Proof. Assume first that we are in the minuscule case. Let $\gamma > 0$ be such that $l(s_\gamma(\pi)) = l(\pi) + 1$. Then, by Lemma 2.16, we have $\langle \varpi - s_\gamma(\pi), \rho^\vee \rangle = \langle \varpi - \pi + \gamma, \rho^\vee \rangle = l(\pi) + 1 = \langle \varpi - \pi, \rho^\vee \rangle + 1$, thus $\langle \gamma, \rho^\vee \rangle = 1$ and γ is a simple root. In the cominuscule case the same argument gives that $\gamma^\vee$ is a simple root, so γ is simple. The coadjoint and adjoint cases are similar, except that one has to take care whether $l(s_\gamma(\pi))$ is divisible by c_1 or not. □

2.5. *Affine symmetries in the adjoint and coadjoint cases*

The description of the quantum monomials in terms of roots gives a nice interpretation of the affine symmetries described in [**10**]. Let us recall the description of these symmetries, we refer to [**10**] for more details. Let c be a cominuscule vertex of the Dynkin diagram and let α_c be the corresponding cominuscule simple root. Denote by τ_c the orientation-preserving automorphism of the extended Dynkin diagram sending the added vertex to the vertex c (there is a unique such automorphism). Denote by $\varpi_c^\vee$ the cominuscule coweight associated to c and by v_c the shortest element such that $v_c\varpi_c^\vee = w_0\varpi_c^\vee$ with w_0 the longest element of the Weyl group. We proved in [**10**] the following formula:

$$\sigma_{v_c} * \sigma_u = q^{\eta_P(\varpi_c^\vee - u^{-1}(\varpi_c^\vee))}\sigma_{v_c u},$$

where $\eta_P : Q^\vee \to Q^\vee/Q_P^\vee$ is the projection from the coroot lattice to its quotient by the coroot lattice of P and where we identify this quotient with $\mathbb{Z}$.

Proposition 2.18. (i) *We have the equality*

$$\sigma_{v_c} = \begin{cases} \sigma_{-\alpha_c - \Theta + \theta} & \textit{in the coadjoint case,} \\ \sigma_{-\alpha_c} & \textit{in the adjoint case.} \end{cases}$$

In particular $\deg(\sigma_{v_c}) = c_1$.

(ii) *For any long real root α in the adjoint case or short real root α in the coadjoint case, we have the formula*

$$\sigma_{v_c} * \sigma_\alpha = \sigma_{\tau_c(\alpha)-\delta} = q\sigma_{\tau_c(\alpha)}.$$

Proof. Point (i) is an easy computation. For (ii), let $u \in W^P$ such that $u(\varpi) = \alpha$. Using the fact that $\tau_c = v_c t_{-\varpi_c^\vee}$, we get $\tau_c(\alpha) = v_c(\alpha) + \langle \alpha, \varpi_c^\vee \rangle \delta$. But we check the equality $\eta_P(\varpi_c^\vee - u^{-1}(\varpi_c^\vee)) = 1 - \langle \alpha, \varpi_c^\vee \rangle$ and the result follows. □

3. *Maximal degree for quantum cohomology*

3.1. *Dimension argument*

In this section we want to bound the possible degrees of the quantum monomials in the quantum parameter q that may appear in a quantum product $\sigma_u * \sigma_v$. The first easy restriction is given by the following.

Lemma 3.1. *Let X be an adjoint or coadjoint homogeneous space. Let u and v be in W^P; then if we write*

$$\sigma_u * \sigma_v = \sum_{d,w} c_{u,v}^w(d) q^d \sigma_w$$

we have $c_{u,v}^w(d) = 0$ unless $d \leqslant 3$.

Proof. This is a simple dimension count. Recall that $c_{u,v}^w(d) = 0$ unless $l(u) + l(v) + \dim X - l(w) = \dim X + c_1(X)d$ thus unless $c_1(X)d \leqslant 2\dim X$. But in the adjoint case, we have $\dim X = 2c_1(X) - 1$ giving $c_{u,v}^w(d) = 0$ unless $d \leqslant 3$. In the coadjoint non-adjoint case, there are three cases. For the Lagrangian Grassmannian, we have $\dim \mathbb{G}_\omega(2, 2n) = 4n - 5 = 2c_1(\mathbb{G}_\omega(2, 2n)) - 3$. Thus we deduce that $c_{u,v}^w(d) = 0$ if $d > 3$. For F_4/P_4 we have $\dim(F_4/P_4) =$

15 and $c_1(F_4/P_4) = 11$, thus $c^w_{u,v}(d) = 0$ unless $d \leqslant 2$. For G_2/P_1, this variety being a quadric, the result is well known (see, for example, [**8**]). □

In the next subsections we show that $c^w_{u,v}(d) = 0$ if $d > 2$. In the coadjoint non-adjoint case, we are in fact more interested in understanding the geometric reason of this vanishing, since the proof of the above lemma almost proves this point (the only case left is that of $\mathbb{G}_\omega(2, 2n)$). Rather what we do is describe in a uniform way the locus Y_3 covered by degree 3 curves passing through two general points (see Definition 3.2) for adjoint and coadjoint varieties and explain that the vanishing of the quantum coefficients is related to a porism: for three points x, y and z in X, if there exists a degree 3 rational curve through these points, then there exist infinitely many such rational curves. We believe that it is worth describing the rich geometry of the subvarieties Y_3.

3.2. *Rational curves on homogeneous spaces*

For the following definition and in all this subsection, we shall only assume that X is a homogeneous space with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$ and we pick the only ample line bundle $\mathcal{L}$ which is a generator of the Picard group (we may also include the adjoint variety for SL_n by taking as ample line bundle $\mathcal{L}$ the line bundle given by its minimal embedding). All degrees are considered with respect to $\mathcal{L}$.

Definition 3.2. Let d be an integer and x and y be points in X.

(i) Let $X_d(x)$ be the locus of points in X connected to x by a rational curve of degree d. In symbols $X_d(x) = \{z \in X \mid \text{there exists } C \text{ a rational degree } d \text{ curve through } x \text{ with } z \in C\}$.

(ii) Let $Y_d(x, y)$ be the locus of points in X connected to x and y by the same rational curve of degree d. In symbols $Y_d(x, y) = \{z \in X \mid \text{there exists } C \text{ a rational degree } d \text{ curve with } x, y, z \in C\}$.

We observed in [**9**] that, for X a minuscule or cominuscule rational homogeneous space, the geometry of rational curves and especially the geometry of the varieties $X_d(x)$ and $Y_d(x, y)$ is intimately related to the quantum cohomology of X. We shall see that this is still the case for adjoint and coadjoint varieties.

Let us denote by $V = V(\varpi)$ the irreducible representation of highest weight ϖ. For π a weight of the representation V, we denote by x_π a vector of weight π. All points in X are equivalent and we therefore will often take x_ϖ, a highest weight vector of V, as a general point in X (recall that X is the orbit of $[x_\varpi]$, the class of x_ϖ in $\mathbb{P}V$).

We introduce the notation $M_{d,3}(X)$ to denote the moduli space of degree d morphism from $\mathbb{P}^1$ to X with three marked points on $\mathbb{P}^1$. Recall from [**32**] or [**26**] that $M_{d,3}(X)$ is irreducible smooth and has the expected dimension $\dim(X) + dc_1(X)$. In particular, we may speak of general degree d rational curves on X. We shall denote by $\mathrm{ev}_i : M_{d,3}(X) \to X$ the evaluation map defined by the ith marked point.

Proposition 3.3. *Let X be a homogeneous space with Picard group* $\mathrm{Pic}(X) = \mathbb{Z}$. *Let x and y be two generic points on a generic curve of degree d; then $X_d(x)$ and $Y_d(x, y)$ are irreducible.*

Proof. We know that $M_{d,3}(X)$ is irreducible. We have a G-equivariant evaluation $\mathrm{ev}_1 : M_{d,3}(X) \to X$. As X is covered by rational curves of any degree, ev_1 is surjective. Assume $x = x_\varpi$. There exists a unipotent subgroup U of G such that the map $U \to X$ defined by

$u \mapsto u \cdot x$ is an isomorphism from U to an open neighbourhood of x, which we denote by O. We have an isomorphism

$$\begin{aligned} \mathrm{ev}_1^{-1}(x_\varpi) \times U &\longrightarrow \mathrm{ev}_1^{-1}(O), \\ (f,u) &\longmapsto u \circ f. \end{aligned}$$

Since $\mathrm{ev}_1^{-1}(O)$, an open subset of $M_{d,3}(X)$, is irreducible, $\mathrm{ev}_1^{-1}(x)$ must be also irreducible. Therefore its image under ev_2, which is $X_d(x_\varpi)$, is irreducible.

For $Y_d(x,y)$ we argue essentially in the same way. We may assume $x = x_\varpi$. We note that $X_d(x_\varpi)$ is a Schubert variety since it is irreducible from the above and B-stable (where B is a Borel subgroup contained in P and containing T) since x_ϖ is fixed by B. Because y and the degree d curve through x_ϖ are supposed to be general, we may assume that y is in the open B-orbit of $X_d(x_\varpi)$. Thus there is again a unipotent subgroup U of B such that $U \to X$ induces an isomorphism between U and an open neighbourhood of y in $X_d(x_\varpi)$. Since U is included in B, it preserves $\mathrm{ev}_1^{-1}(x_\varpi)$, and thus induces an isomorphism $\mathrm{ev}_1^{-1}(x_\varpi) \simeq (\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(x_\varpi, y) \times U$. Therefore $(\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(x_\varpi, y)$ is irreducible and we conclude as before. □

Note that the proof shows that $Y_d(x,y)$ does not depend, up to isomorphism, on the choice of the two points x and y on a generic curve of degree d. Thus we can introduce the following.

DEFINITION 3.4. Let $X_d(X)$ or $Y_d(X)$ be the class of the varieties $X_d(x)$ or $Y_d(x,y)$ modulo linear isomorphisms for one or two generic point(s) x or x and y on a generic degree d curve, respectively. When no confusion on the variety X can be made, we simplify $X_d(X)$ and $Y_d(X)$ in X_d and Y_d, respectively.

COROLLARY 3.5. *Let X be a homogeneous space with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$. The variety X_d in X is a Schubert variety stable under P.*

DEFINITION 3.6. (i) We shall denote by $x_\varpi(d)$ the T-stable point in the open B-orbit of the Schubert variety $X_d(x_\varpi)$.

(ii) For d and n two non-negative integers, we denote by $\delta_X(d,n)$ the dimension of the space of degree d rational curves passing through n generic points of a generic curve of degree d in X.

We now prove a formula involving the dimension of X_d and of Y_d.

PROPOSITION 3.7. *Let X be a homogeneous space with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$; then we have*

$$\dim(X_d) + \dim(Y_d) + \delta_X(d,3) = c_1 d.$$

Proof. The moduli space $M_{d,3}(G/P)$ has dimension $\dim(X) + dc_1$. Therefore, as X is covered by rational curves of any degree, the inverse image $\mathrm{ev}_1^{-1}(x_\varpi)$ has dimension dc_1. Considering the morphism $\mathrm{ev}_2 : \mathrm{ev}_1^{-1}(x_\varpi) \to X$, with image $X_d(x_\varpi)$, we deduce the following equality of dimensions: $\dim((\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(x_\varpi, x_\varpi(d)) = c_1 d - \dim(X_d)$.

Now the image of $\mathrm{ev}_3 : (\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(x_\varpi, x_\varpi(d)) \to X$ is by definition $Y_d(x,y)$, and the fibres of this morphism are $\delta_X(d,3)$-dimensional. □

We recall here how to describe chains of T-invariant rational curves (see, for example, [**16**]).

PROPOSITION 3.8. *Let X be a homogeneous space with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$. The chains of k irreducible T-invariant rational curves starting from x_ϖ of degrees $(d_i)_{i\in[1,k]}$ are in bijection with sequences $(\alpha_i)_{i\in[1,k]}$ of positive roots with $\langle s_{\alpha_{m-1}} \dots s_{\alpha_1}(\varpi), \alpha_m^\vee\rangle = d_m$ for all m. The end point of the mth component of such a chain is the projectivization of the weight line of weight $s_{\alpha_m} \dots s_{\alpha_1}(\varpi) = \varpi - d_1\alpha_1 - \dots - d_m\alpha_m$.*

We shall need the following.

LEMMA 3.9. *Let X be a homogeneous space with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$. Let $x, y \in X$ be two T-invariant points and d be an integer. Assume that there is exactly one T-invariant chain of degree d through x and y. Then there is only one chain of degree d through x and y.*

Proof. Assume that there is a chain which is not T-invariant. Then the T-orbit of this chain provides a positive-dimensional family of chains of degree d through x and y. In the moduli space of curves, the corresponding positive-dimensional variety must have at least 2 fixed points, which is a contradiction. □

3.3. *Adjoint case*

We first investigate the geometry of conics in adjoint homogeneous spaces, so we fix an adjoint homogeneous space X.

PROPOSITION 3.10. *We have $X_2 = X$ and Y_2 is a conic.*

Proof. Recall that Θ denotes the highest root. Since $\langle\Theta, \Theta\rangle = 2$, the curve through x_Θ with tangent direction $-\Theta$ has degree 2; its other T-fixed point has weight $s_\Theta(\Theta) = -\Theta$, thus it is $x_{-\Theta}$, that is, the T-fixed point representing the dense orbit. Therefore $X_2 = X$.

To understand Y_2, we determine the degree 2 T-invariant curves through x_Θ and $x_{-\Theta}$. By Proposition 3.8, such a curve is either given by a positive root α such that $\langle\Theta, \alpha\rangle = 2$ and $\Theta - 2\alpha = -\Theta$ (in which case $\alpha = \Theta$: this corresponds to the previous conic), or by a sequence (β, α) of two positive roots such that $\langle\Theta, \alpha\rangle = \langle s_\alpha(\Theta), \beta\rangle = 1$ and $\Theta - \alpha - \beta = -\Theta$. Since Θ is the highest root, this last equation implies $\alpha = \beta = \Theta$, contradicting the degrees.

Thus there is only one T-invariant degree 2 curve through x_Θ and $x_{-\Theta}$. By Lemma 3.9 there is only one degree 2 curve through x_Θ and $x_{-\Theta}$ and this implies that Y_2 is a conic. □

DEFINITION 3.11. We denote by $Y_3' \subset X$ the union of all the lines in X that meet Y_2.

PROPOSITION 3.12. *Let x, y, z be three points in X. If there is a curve of degree 3 through x, y, z, then there are infinitely many. Thus $Y_3 = Y_3'$.*

Proof. Let $M_{d,3}(x, y)$ denote the inverse image $(\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(x, y)$. The proof of Proposition 3.3 shows that $M_{3,3}(x, y)$ is irreducible of dimension $3c_1 - \dim X = c_1 + 1$.

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc} M_{3,3}(x,y) & \xrightarrow{\mathrm{ev}_3} & Y_3 \\ \uparrow & & \uparrow \\ \mathrm{ev}_3^{-1}(Y_3')\cap M_{3,3}(x,y) & \xrightarrow{\mathrm{ev}_3} & Y_3'. \end{array}$$

Therefore, if we prove that $\dim Y_3'$ is equal to c_1 and that the dimension of the general fibre of the second horizontal map is equal to one, this would imply the following dimension equalities:

$$\dim(\mathrm{ev}_3^{-1}(Y_3')\cap M_{3,3}(x,y)) = c_1 + 1 = \dim M_{3,3}(x,y),$$

therefore $M_{3,3}(x,y) \subset \mathrm{ev}_3^{-1}(Y_3')$ and the result would follow.

Since the dimension of the space of curves through three points is an upper-continuous function on the three points, we may assume that x and y are generic, thus that $x = x_\Theta$ and $y = x_{-\Theta}$. Since the dimension of the variety of lines through any point in X is $c_1 - 2$, the dimension of Y_3' is at most c_1. It is in fact c_1 since T_xX and T_yX only meet along the projectivization of the line $[\mathfrak{g}_\Theta, \mathfrak{g}_{-\Theta}]$ in $\mathfrak{g}$.

To prove that $Y_3(x_\Theta, x_{-\Theta}) = Y_3'$, we need to prove that the general fibre of the second horizontal map in the previous diagram has dimension 1. For this we use a suitable decomposition of $\mathfrak{g}$. Let $L \subset G$ be the pointwise stabilizer of $C = Y_2(x_\Theta, x_{-\Theta})$ and $\mathfrak{l}$ be its Lie algebra. If α is a root of $\mathfrak{l}$, then $\langle\Theta, \alpha^\vee\rangle = 0$. Let $\mathfrak{sl}_2(\Theta)$ be generated by $\mathfrak{g}_\Theta$, $\mathfrak{g}_{-\Theta}$ and $[\mathfrak{g}_\Theta, \mathfrak{g}_{-\Theta}]$. We consider $\mathfrak{g}$ as a module over $\mathfrak{sl}_2 \times \mathfrak{l}$. This module certainly contains $\mathfrak{sl}_2 \times \mathfrak{l}$. The tangent space T_xX of X at x or y can be identified with the T-stable subspace of $\mathfrak{g}$ with weights the negative roots α such that $\langle\Theta, \alpha^\vee\rangle < 0$ or the positive roots α such that $\langle\Theta, \alpha^\vee\rangle > 0$, respectively. We define the subspace T_x^1X of T_xX or T_y^1X of T_yX as the T-stable subspace with weights the negative roots α such that $\langle\Theta, \alpha^\vee\rangle = -1$ or the positive roots α such that $\langle\Theta, \alpha^\vee\rangle = 1$, respectively. Since $L \subset P$, the space $V_L := T_x^1X$ is a sub-L-module of $\mathfrak{g}$. Similarly T_y^1X is also sub-L-module of $\mathfrak{g}$, and moreover T_x^1X and T_y^1X are disjoint. Counting dimensions, we see that we have

$$\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{l} \oplus \mathbb{C}^2 \otimes V_L$$

Concretely, here is the decomposition in each case:

$\mathfrak{g}$	Decomposition of $\mathfrak{g}$
$\mathfrak{sl}_n$	$(\mathfrak{gl}_2 \times \mathfrak{gl}_{n-2})_0 \oplus \mathbb{C}^2 \otimes (\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*)$
$\mathfrak{so}_{2n+1}$	$\mathfrak{sl}_2 \times (\mathfrak{sl}_2 \times \mathfrak{so}_{2n-3}) \oplus \mathbb{C}^2 \otimes (\mathbb{C}^2 \otimes \mathbb{C}^{2n-3})$
$\mathfrak{sp}_{2n}$	$\mathfrak{sl}_2 \times \mathfrak{sp}_{2n-2} \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2n-2}$
$\mathfrak{so}_{2n}$	$\mathfrak{sl}_2 \times (\mathfrak{sl}_2 \times \mathfrak{so}_{2n-4}) \oplus \mathbb{C}^2 \otimes (\mathbb{C}^2 \otimes \mathbb{C}^{2n-4})$
$\mathfrak{e}_6$	$\mathfrak{sl}_2 \times \mathfrak{sl}_6 \oplus \mathbb{C}^2 \otimes \wedge^3\mathbb{C}^6$
$\mathfrak{e}_7$	$\mathfrak{sl}_2 \times \mathfrak{so}_{12} \oplus \mathbb{C}^2 \otimes V^{32}$
$\mathfrak{e}_8$	$\mathfrak{sl}_2 \times \mathfrak{e}_7 \oplus \mathbb{C}^2 \otimes V^{56}$
$\mathfrak{f}_4$	$\mathfrak{sl}_2 \times \mathfrak{sp}_6 \oplus \mathbb{C}^2 \otimes \wedge^3_\omega\mathbb{C}^6$
$\mathfrak{g}_2$	$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \oplus \mathbb{C}^2 \otimes S^3\mathbb{C}^2$

Using this description, we can show that $Y_3 = Y_3'$. Let $\nu : \mathbb{C}^2 \to \mathfrak{sl}_2$ be the natural map of degree 2 with image the affine cone over $\mathfrak{sl}_2^{\mathrm{ad}}$, the adjoint variety for $\mathfrak{sl}_2$. Let $X_L \subset V_L$ be the affine cone over the closed L-orbit in $\mathbb{P}V_L$. Then Y_3' is described as the set of classes of elements $(s, 0, v \otimes \alpha) \in \mathfrak{sl}_2 \times \mathfrak{l} \oplus \mathbb{C}^2 \otimes V_L$, with $\alpha \in X_L$, and s and $\nu(v)$ proportional. The points x and y are $[\nu(1,0) : 0 : 0]$ and $[\nu(0,1) : 0 : 0]$, respectively.

If $z = [\nu(v), 0, v \otimes \alpha] \in Y_3' \subset X$, $q : \mathbb{C}^2 \to \mathbb{C}$ is the quadratic form vanishing on $(1,0)$ and $(0,1)$ and such that $q(v) = 1$, l is a linear form such that $l(v) = 1$, then the map

$$\begin{aligned} \mathbb{P}^1 = \mathbb{P}\mathbb{C}^2 &\longrightarrow Y_3', \\ [w] &\longmapsto [l(w)\nu(w) : 0 : q(w)w \otimes \alpha] \end{aligned}$$

is a rational curve of degree 3 through x, y and z. Thus there are infinitely many rational curves of degree 3 through x, y, z, so $Y_3' = Y_3$ and the proposition is proved. □

3.4. *Coadjoint varieties as hyperplane sections of Scorza varieties*

Scorza varieties were first defined and classified by Zak as varieties satisfying an extremal property with respect to their higher secants (see [**33**]). For our purpose it will be enough to use the following classification theorem as a definition.

Definition 3.13. A Scorza variety is defined by two integers δ and m, with $\delta \in \{2, 4, 8\}$ and $m \geqslant 3$. If $\delta = 8$, then we must have $m = 3$. The corresponding Scorza varieties $X(\delta, m)$ are as follows (note that Scorza varieties are homogeneous and that a generic hyperplane section of a Scorza variety turns out to be itself homogeneous; this homogeneous space is described in the third column):

δ	$X(\delta, m)$	$X(\delta, m) \cap H$
2	$\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$	$\mathbb{F}(1, m-1, m)$
4	$\mathbb{G}(2, 2m)$	$\mathbb{G}_\omega(2, 2m)$
8	E_6/P_1	F_4/P_4

The reason why we consider Scorza varieties is that we recover the coadjoint non-adjoint varieties as hyperplane sections of some Scorza varieties (except for G_2/P_1). Actually Zak's definition of Scorza varieties allows $\delta = 1$, but we do not need the varieties $X(1, m)$ and therefore will not speak about them.

Those varieties are naturally embedded via the smallest ample divisor $\mathcal{O}(1)$. The numbers δ, m have some geometric and algebraic meanings: δ is the secant defect of $X(\delta, m)$, meaning that $\dim(\mathrm{Sec}(X)) = 2\dim(X) + 1 - \delta$, where $\mathrm{Sec}(X)$ denotes the Zariski closure of the union of the secant lines to X. More precisely through two generic points of X there passes a unique δ-dimensional quadric. Moreover, m is the smallest integer such that the mth higher secant of X is the whole projective space [**33**].

On the other hand, the vector space in the projectivization of which $X(\delta, m)$ is embedded has a natural structure of a Jordan algebra, the integer m is the generic rank of this algebra and δ the dimension of the composition algebra which coordinatises this algebra [**17**].

Lemma 3.14. *We have* $c_1(X) = (\delta/2)m$ *and* $\dim(X) = \delta(m-1)$.

Proof. The following numbers are well known

X	$\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$	$\mathbb{G}(2, 2m)$	E_6/P_1
δ	2	4	8
$\dim(X)$	$2(m-1)$	$2(2m-2)$	16
$c_1(X)$	m	$2m$	12

and yield the lemma. □

3.5. *A description of Y'_3 for Scorza varieties*

In this subsection, we fix X a Scorza variety with parameters (δ, m). We first describe Y_2.

PROPOSITION 3.15. *Through two generic points in X there passes a conic, and Y_2 is a smooth quadric of dimension δ.*

Proof. Let $x, y \in X$ be generic. By [**33**], through x and y there passes a unique quadric of dimension δ, which we denote by $Q_{x,y}$. This proves the first claim of the lemma. Moreover, we have $Q_{x,y} \subset Y_2(x, y)$. Since $Y_2(x, y)$ is irreducible by Proposition 3.3, and has dimension at most $2c_1(X) - \dim(X) = \delta$ by Lemma 3.14 and Proposition 3.7, the second claim follows. □

DEFINITION 3.16. (i) Let $d > 0$, $e \geqslant 0$ be integers, Spin_{2d} be the corresponding spinor group; let A and $\mathbb{S}$ be the vector and a spinor representation, with highest weight vector a_0 and s_0, respectively. We denote by $S \subset \mathbb{PS}$ the spinor variety. Let W denote the natural representation of SL_e with highest weight vector w_0.

(ii) We denote by $S(d, e)$ the variety that is the closure of the $(\mathrm{Spin}_{2d} \times \mathrm{SL}_e)$-orbit of $[a_0 + s_0 \otimes w_0]$ in $\mathbb{P}(A \oplus (S \otimes W))$. The closed orbit of $[a_0]$, which is a smooth quadric of dimension $2d - 2$, is a closed subvariety of $S(d, e)$ denoted by $Q(d)$.

REMARK 3.17. We will prove that the varieties Y'_3 are projectively equivalent to varieties of the form $S(d, e)$. Note that the varieties $S(d, e)$ are horospherical. In all the examples that we were able to compute, the varieties covered by rational curves of a fixed degree through two fixed points in a homogeneous space of Picard rank 1 are horospherical. It would be interesting to know if this is always the case. For more details on horospherical varieties we refer, for example, to [**25**].

LEMMA 3.18. *The variety $S(d, e)$ has dimension $((d + 2)(d - 1))/2 + e$. Any point in $S(d, e) - Q(d)$ is contained in a linear space of dimension d included in $S(d, e)$, and intersecting $Q(d)$ along a maximal isotropic subspace (of dimension $d - 1$).*

Proof. Let $x = [a + s \otimes w]$ be a generic point in $S(d, e)$, with $a \in A, s \in \mathbb{S}, w \in W$ and $s \otimes w \neq 0$. Since x is in the $\mathrm{Spin}_{2d} \times \mathrm{SL}_e$-orbit of $[a_0 + s_0 \otimes w_0]$, the class $[s]$ of s in $\mathbb{PS}$ it is a point of the spinor variety, thus it corresponds to a maximal isotropic linear subspace $L_d(s) \subset A$, and a is isotropic. Moreover, $a \in L_d(s)$. Since the stabilizer in Spin_{2d} of an element $[s] \in S$ acts transitively on the projectivization of $L_d(s)$, it follows that $S(d, e)$ is the variety of all classes $[a' + s' \otimes w']$ satisfying $[s'] \in S$ and $a' \in L_d(s')$. It has dimension $d(d - 1)/2 + (d - 1) + (e - 1) + 1$. Since $x \in S(d, e)$, it follows that $a \in L_d(s)$. Thus the linear space $\mathbb{P}(L_d(s) \oplus \mathbb{C} \cdot s \otimes w)$ satisfies the conditions of the lemma. □

Recall that we denote by ϖ the dominant weight corresponding to X. For X a Scorza variety, remark that ϖ is minuscule. We define two subgroups of G as follows. Let P be the parabolic subgroup of G such that $G/P = X$ and denote by L a Levi subgroup. The Dynkin diagram of L can be recovered from the Dynkin diagram of G by removing the vertex v corresponding to P and the vertex $i(v)$, where i is the Weyl involution.

Definition 3.19. (i) We denote by L_1 the subgroup of L whose Dynkin diagram is the union of the connected components of the Dynkin diagram of L non-orthogonal to Θ the highest root of G, or equivalently connected to the added vertex in the extended Dynkin diagram of G.

(ii) We denote by L_2 the subgroup of L whose Dynkin diagram is the union of the connected components of the Dynkin diagram of L orthogonal to Θ the highest root of G, or equivalently non-connected to the added vertex in the extended Dynkin diagram of G.

(iii) We denote by $\tilde{L}_1$ the subgroup of G generated by L_1 and $\mathrm{SL}_2(\Theta)$ (the subgroup of G whose Lie algebra is $\mathfrak{g}_{-\Theta} + [\mathfrak{g}_{-\Theta}, \mathfrak{g}_{\Theta}] + \mathfrak{g}_{\Theta}$). Its Dynkin diagram is the Dynkin diagram of L_1 together with the added vertex in the extended Dynkin diagram of G.

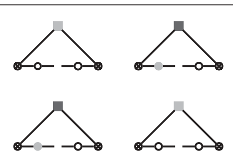
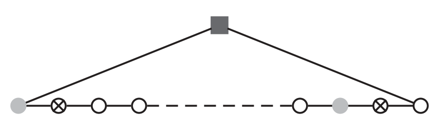
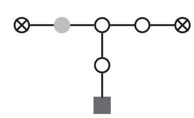
Pictures of the Dynkin diagrams of G, L, $\tilde{L}_1$ and L_2 for Scorza varieties are given in the proof of the following.

Proposition 3.20. *The variety* $Y_3'(E_6/P_1)$ *or* $Y_3'(\mathbb{G}(2, 2m))$ *is projectively equivalent to* $S(5, 1)$ *or* $S(3, 2m-4)$*, respectively. The variety* $Y_3'(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1})$ *is a union of two irreducible components equivalent to* $S(2, m-2) \simeq \mathbb{P}^1 \times \mathbb{P}^{m-1}$ *and meeting along a smooth quadric surface.*

Proof. Let α be a maximal root with vanishing pairing with $i(\varpi)$. In the case of E_6/P_1, $\mathbb{G}(2, 2m)$ or $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, we have, respectively,

$$\alpha = \begin{pmatrix} 1\,2\,2\,1\,0 \\ 1 \end{pmatrix}, \quad \alpha = (1, 1, \ldots, 1, 0, 0) \quad \text{or} \quad \alpha = \begin{cases} ((1, \ldots, 1, 0), (0, \ldots, 0)) \text{ or} \\ ((0, \ldots, 0), (1, \ldots, 1, 0)). \end{cases}$$

Note that we have $\langle \Theta^\vee, \alpha \rangle = 1$, thus $\langle -\Theta^\vee, \varpi - \alpha \rangle = 0$. We let $v_1 \in V$ be a weight vector of weight ϖ, and $v_d \in V$ of weight $s_\alpha(\varpi) = \varpi - \alpha$. It follows that the line through v_1 and v_d is included in X, and, moreover, we see that v_1 is the lowest weight of the vector representation of $\tilde{L}_1 \simeq \mathrm{Spin}_{\delta+2}$ and v_d is the lowest weight of the representation $\mathbb{S} \otimes W_e$, where $\mathbb{S}$ is a spin representation of $\tilde{L}_1$ and W_e is the natural representation of a simple factor SL_e of the group L_2 (see the following array: the black circles represent the Dynkin diagram, one has to remove the crossed ones to find L_1; the dark grey squares represent the longest root and the dark grey discs the weight $\varpi - \alpha$).

X	$\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$	$\mathbb{G}(2, 2m)$	E_6/P_1
			
d	2	3	5
e	$m-2$	$2m-4$	1

It follows that Y_3' contains the closure S of the $(\tilde{L}_1 \times L_2)$-orbit of $[v_1 + v_d]$, which is by definition equivalent to an $S(d, e)$. The equality $Y_3' = S$ follows counting dimensions. In fact, observe from the above array that we always have $e = ((d-1)/2)(2m-d) - 1$ (we have no better way to check this than making the computation in the three cases $d = 2, 3, 5$). This may be rewritten as

$$\frac{(d+4)(d-1)}{2} + e - 1 = (d-1)m + 2d - 4. \tag{2}$$

We may find lines in S which meet Y_2 by choosing any point in S and then a line in the $\mathbb{P}^d$ which contains it and was described in Lemma 3.18, taking into account the fact that in this way each line is counted ∞^1 times. The dimension of the set of these lines is thus the left-hand side of (2) since $\dim S = ((d+2)(d-1))/2 + e$ by Lemma 3.18. On the other hand the variety of lines in X passing through a point in Y_2 has dimension $\dim\ Y_2 + c_1(X) - 2$, which is the right-hand side of (2) by Lemma 3.14 since $\delta = 2d - 2$. Thus it follows that any line meeting Y_2 is included in S, so that $S = Y_3'$. □

Remark 3.21. There is a uniform description of $Y_3'(X)$: in all cases $Y_3'(X)$ is the linear section $X \cap \mathbb{P}(T_P \operatorname{Sec}(X))$, where P is a generic point in the span of Y_2. For a proof of this fact, see [**14**, Proposition 3.20].

3.6. Y_3' *for a hyperplane section of a Scorza variety*

Let X be a Scorza variety and W be a general hyperplane section of X. We reduce the computation of $Y_3(W)$ to the case when $X = \mathbb{P}^1 \times \mathbb{P}^2$ (which is not a Scorza variety), and therefore we start studying this case.

Lemma 3.22. *Let $X = \mathbb{P}^1 \times \mathbb{P}^2$ and W be a generic hyperplane section of X. Given three points x_1, x_2, x_3 in W and a reducible curve R of degree 3 through these points, R is a limit of some irreducible curves passing through x_1, x_2, x_3 and contained in W.*

Thus the same result holds for a generic curve in X.

Proof. First of all, let us describe W as a scroll. Restricting the linear form H to any $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^2$ yields a linear map $\mathbb{C}^3 \to \mathbb{C}^2$, with 1-dimensional kernel. Let $x \in \mathbb{P}^2$ correspond to this kernel: it is the only element in $\mathbb{P}^2$ such that $\mathbb{P}^1 \times \{x\}$ is contained in H. Take a line l in $\mathbb{P}^2$ not containing x and let $C := (\mathbb{P}^1 \times l) \cap H$. Then C is a conic which may be described as the set of elements $(a, f(a))$ for some isomorphism $f : \mathbb{P}^1 \to l$. Then W is the union of all the lines joining $(a, f(a)) \in C$ and $(a, x) \in \mathbb{P}^1 \times \{x\}$.

Now, given three generic points x_1, x_2, x_3 on W, we may choose C such that $x_1, x_2 \in C$, so that $x_i = (a_i, f(a_i)) \in \mathbb{P}^1 \times l$ for some $a_i \in \mathbb{P}^1$ and $i = 1, 2$. Then the third point is on the line joining a point $(a_3, f(a_3))$ in C and $(a_3, x) \in \mathbb{P}^1 \times \{x\}$. Thus we may assume $R = C \cup l$, where l is the line between $(a_3, f(a_3))$ and (a_3, x).

If h or q is a linear or quadratic form on $\mathbb{P}^1$, respectively, then the morphism

$$t \mapsto (t, h(t)f(t) + q(t)x) \in W \subset \mathbb{P}^1 \times \mathbb{P}^2$$

will pass through x_1, x_2 provided that q vanishes at a_1, a_2, and imposing, moreover, that it passes through x_3 yields one more relation between h and q. Thus there are indeed infinitely many irreducible curves of degree 3 through x_1, x_2, x_3.

If q tends to 0 with the vanishing point of h tending accordingly to a_3, we see that the limit of those curves must contain C. Since it is connected and contains x_3, this limit can only be R. □

Let X be a Scorza variety and $W = X \cap H$ be a generic hyperplane section of X.

Proposition 3.23. *We have $Y_3(X) = Y_3'(X)$ and $Y_3(W) = Y_3(X) \cap H = Y_3'(W)$, and if there passes a curve of degree 3 through three points of W, then there pass infinitely many such curves.*

Proof. We show that any reducible curve R in X or W of degree 3 through x_1, x_2, x_3 is a limit of some irreducible curves passing through the same points, which implies the proposition.

We first consider the case of X. To see this, we use the description of $Y_3'(X)$ given in Proposition 3.20. By an action of G, we may assume that x_1 or x_2 is the projectivization of the highest or lowest weight line in V, respectively, so that $x_1 = a_1, x_2 = a_2$ are some points on the quadric $Q(d)$, and x_3 is on the line joining a point a_3 in $Q(d)$ and a point $b_3 = s_3 \otimes w_3$ in $S \times \mathbb{P}W$, with $a_3 \in L_d(s_3)$, and finally that R is the union of this line and a conic C included in $Q(d)$ and passing through a_1 and a_2. Let $V_{d-2} = L_d(s_3) \cap \langle a_1, a_2, a_3\rangle^\perp$ be an isotropic subspace of dimension $d-2$. We choose a non-degenerate subspace V_4 of dimension 4 included in $V_{d-2}^\perp$, containing a_1, a_2, a_3, and disjoint from V_{d-2}.

Since $\mathbb{P}V_4 \cap Q(d)$ is a smooth quadric surface, to each point $a \in \mathbb{P}V_4 \cap Q(d)$ corresponds an isotropic plane that we denote by V_a, such that $\tilde{V}_a := V_{d-2} \oplus V_a$ is a maximal isotropic subspace in V_{2d} parametrized by a point s_a in S. Remark that any two distinct subspaces $\tilde{V}_a, \tilde{V}_b$ meet in codimension 2 (namely, along V_{d-2}), so that the curve traced out by the points s_α in the spinor variety is in fact a line.

It follows that X contains a $\mathbb{P}^1 \times \mathbb{P}^2$, namely the union of all the planes $\mathbb{P}(V_a \oplus \mathbb{C}.s_a \otimes w_3)$ for a in the quadric surface. Moreover, since $\tilde{V}_{a_3} = L_d(s_3)$, this $\mathbb{P}^1 \times \mathbb{P}^2$ contains the line joining a_3 and b_3. Since $\mathbb{P}V_4 \cap Q(d)$ contains a_1, a_2, a_3, it contains C and thus our $\mathbb{P}^1 \times \mathbb{P}^2$ contains our initial curve R. Thus, by Lemma 3.22, R can be deformed to an irreducible curve passing through x_1, x_2, x_3.

If we assume from the beginning that R is included in $W = X \cap H$, then again from Lemma 3.22, the same conclusion holds. □

THEOREM 4. *Let X be an adjoint or a coadjoint variety and let σ_u and σ_v be classical cohomology classes. Then if we write*

$$\sigma_u * \sigma_v = \sum_{d,w} c_{u,v}^w(d) q^d \sigma_w$$

we have $c_{u,v}^w(d) = 0$ unless $d \leqslant 2$.

Proof. First, by Proposition 3.12 and Proposition 3.23, we know that $c_{u,v}^w(3) = 0$. Lemma 3.1 concludes the proof for $c_{u,v}^w(d)$ with $d > 3$. □

COROLLARY 3.24. *Let X be an adjoint variety; then in the quantum cohomology $QH^*(X, \mathbb{C})$ we have the formula*

$$\mathrm{pt} * \mathrm{pt} = 2q^2\ell,$$

where pt *or ℓ denotes the class of a point or the class of a line in X, respectively.*

Proof. By degree counting the first quantum terms are of degree 2 in q. Furthermore, by the previous result, there are no higher powers of q. We thus have to prove the equality $I_2(\mathrm{pt}, \mathrm{pt}, h) = 2$ where h is the hyperplane class. As we have seen in Proposition 3.10, there is a unique conic through two general points in X. As this conic meets a hyperplane in two points, the result follows. □

4. *Quantum to classical principle for degree* 1 *Gromov–Witten invariants*

In this section, we explain how to apply by now classical techniques initiated by Buch, Kresch and Tamvakis [5] to compute degree 1 Gromov–Witten invariants. Here we shall use the presentation of [9]. Most of the arguments are very similar to those of [9, Section 3] and we shall therefore not reproduce them.

Let F be the variety of lines on X. This variety is well described in [23]. In particular, note that this variety is homogeneous for X adjoint but is not homogeneous for X coadjoint and not adjoint. In the last case, the comparison principle is less useful to compute Gromov–Witten invariants on X than to compute intersections on F.

Let us recall some notation from [9]. Points in F are denoted by ℓ and we shall denote by L the associated line in X. We have an incidence $I = \{(x,\ell) \in X \times F / x \in L\}$ between X and F. We denote by p or q the projection from I to X or F, respectively. For $w \in W^P$, we denote the variety $q(p^{-1}(X(w)))$ by $F(\widehat{w})$.

The following lemma is an adaptation of [9, Lemma 3.7] to the non-minuscule case. For $w \in W^P$, we denote by $w^\vee$ the element of W^P such that $X(w)$ and $X(w^\vee)$ are Poincaré dual.

Lemma 4.1. *Let $X(w)$ and $X(v^\vee)$ be two Schubert varieties such that $X(w) \subset X(v^\vee)$. Then there exists an element $g \in G$ such that the intersection $X(w) \cap g \cdot X(v)$ is a reduced point.*

Proof. There always exists a sequence of inclusions of Schubert varieties

$$X(w) = X(w_0) \subset X(w_1) \subset \ldots \subset X(w_r) = X(v^\vee),$$

where $X(w_i)$ is a moving divisor in $X(w_{i+1})$ for all i. We prove, by descending induction on $i \in [0, r]$, that there exists an element g_i in G such that the intersection $X(w_i) \cap g_i \cdot X(v)$ is a reduced point. For $i = r$ this is Poincaré duality. Assume that the result holds for w_{i+1}, that is, there exists an element $g_{i+1} \in G$ such that $X(w_{i+1}) \cap g_{i+1} \cdot X(v)$ is a reduced point say x. Recall that because $X(w_i)$ is a moving divisor in $X(w_{i+1})$, we have the equality

$$X(w_{i+1}) = \bigcup_{h \in \mathrm{Stab}(X(w_{i+1}))} h \cdot X(w_i).$$

We deduce from this that there exists an element $h \in \mathrm{Stab}(X(w_{i+1}))$ such that $x \in h \cdot X(w_i)$. But now $h \cdot X(w_i)$ meets $g_{i+1} \cdot X(v)$ in x at least. Since $h \cdot X(w_i)$ is contained in $X(w_{i+1})$ and $X(w_{i+1}) \cap g_{i+1} \cdot X(v)$ is the reduced point x, we have that $h \cdot X(w_i)$ and $g_{i+1} \cdot X(v)$ meet transversely and only in x. □

For the next two results, the proofs of [9, Lemma 3.10 and Corollary 3.11] apply verbatim.

Lemma 4.2. *Let $X(u)$, $X(v)$ and $X(w)$ be three proper Schubert subvarieties of X, such that*

$$\mathrm{Codim}(X(u)) + \mathrm{Codim}(X(v)) + \mathrm{Codim}(X(w)) = \dim(X) + c_1(X).$$

Then, for g, g' and g'' three general elements in G, the intersection $g \cdot F(\widehat{u}) \cap g' \cdot F(\widehat{v}) \cap g'' \cdot F(\widehat{w})$ is a finite set of reduced points.

Let ℓ be a point in this intersection; then the line L meets each of $g \cdot X(u)$, $g' \cdot X(v)$ and $g'' \cdot X(w)$ in a unique point and these points are in general position in L.

Corollary 4.3. *Let $X(u)$, $X(v)$ and $X(w)$ be three proper Schubert subvarieties of X. Suppose that the sum of their codimensions is $\dim(X) + c_1(X)$. Then*

$$I_1([X(u)], [X(v)], [X(w)]) = I_0([F(\widehat{u})], [F(\widehat{v})], [F(\widehat{w})]).$$

As in [**9**], we may use this result to compute combinatorially some Gromov–Witten invariants. Let us simply give an example of this.

EXAMPLE 4.4. We assume that X is of adjoint type and homogeneous under a group of type different from A_n. Let u and v be elements in W^P such that $l(u) + l(v) = c_1(X) = \deg(q)$. We want to compute the quantum product $\sigma^u * \sigma^v$.

Let us first remark that in this situation, the only non-classical term appearing in $\sigma^u * \sigma^v$ is of the form aq for some $a \in \mathbb{Z}$. In symbols

$$\sigma^u * \sigma^v = \sigma^u \cup \sigma^v + aq.$$

The integer a is equal to the Gromov–Witten invariant $I_1([X(u^\vee)], [X(v^\vee)], [\mathrm{pt}])$. By the previous discussion, this invariant is equal to $I_0([q(p^{-1}(X(u^\vee)))], [q(p^{-1}(X(u^\vee)))], [q(p^{-1}(\mathrm{pt}))])$.

In our situation, the variety F is homogeneous of the form G/Q for Q a parabolic subgroup. By the same discussion as in [**9**, Subsection 3.3], the variety $F(\widehat{w}) = q(p^{-1}(X(w)))$ for $X(w)$ a Schubert variety in X is a Schubert variety in F. The notation $\widehat{w}$ will in that situation stand for the corresponding element in W^Q. Furthermore, the incidence variety I is also homogeneous of the form G/R with $R = P \cap Q$ and the inverse image $p^{-1}(X(w))$ of a Schubert variety is again a Schubert variety. We may obtain the element $\widehat{w}$ easily from w by the following procedure.

(i) First remark that P is maximal associated to a simple root, say α, and that w_X the longest element in W^P has the form $s_\alpha \widetilde{w}_X$ with $l(\widetilde{w}_X) = l(w_X) - 1$. In other words $\widetilde{w}_X = s_\alpha w_X$.

(ii) Let Z denote the fibre $p^{-1}(\mathrm{pt})$ of the 0-dimensional Schubert variety pt in X. This is a Schubert variety in I and let $w_Z \in W^R$ be the corresponding element. Then $p^{-1}(X(w))$ is the Schubert variety in I associated to $ww_Z \in W^R$.

(iii) If $w \neq w_X$, then we have $\widehat{w} = ww_Z \in W^Q$. We have $\widehat{w}_X = s_\alpha w_X w_Z$. This last element is the longest element in W^R; we denote it by w_F. This element is self-inverse, thus $w_F = w_Z w_X s_\alpha$.

We want to compute the Littlewood–Richardson coefficient

$$I_0([F(\widehat{u^\vee})], [F(\widehat{v^\vee})], [F(w_Z)]) = c^{w_Z}_{\widehat{u^\vee}^\star, \widehat{v^\vee}^\star}$$

in F with $\star$ denoting Poincaré duality in F. Now compute, for $l(u) > 0$,

$$\widehat{u^\vee}^\star = w_0 \widehat{u^\vee} w_0 w_F = w_0 (u^\vee w_Z) w_0 w_F = w_0 (w_0 u w_0 w_X) w_Z w_0 w_F = u w_X w_Z w_F = u s_\alpha.$$

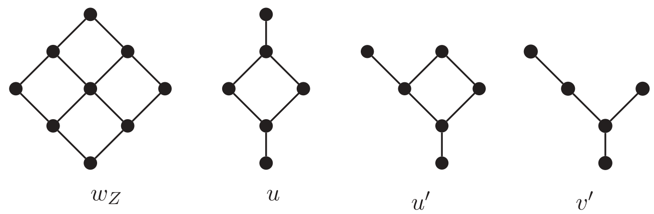
The same is true for $\widehat{v^\vee}^\star$. We thus get the following proposition.

PROPOSITION 4.5. *Let X be of adjoint type and assume that G does not have type A, and let $u, v \in W^P$ such that $l(u) + l(v) = c_1(X)$. Then*

$$\sigma^u * \sigma^v = \sigma^u \cup \sigma^v + c^{w_Z}_{us_\alpha, vs_\alpha} \cdot q$$

Remark that for X adjoint the fibre Z is always a product of minuscule and cominuscule homogeneous spaces. In particular if Λ is the weight associated to Q, then w_Z is Λ-minuscule and we may compute the above Littlewood–Richardson coefficient using jeu de taquin by [**12**, Theorem 1.3] (and, moreover, in this case, jeu de taquin is trivial because we multiply by the fundamental class of the variety F and we therefore only need to understand Poincaré duality in F).

Let us do an explicit example. We refer to [**12**] for more details on jeu de taquin and the notion of heap. Let $X = E_6/P_2$, the heap associated to w_Z, be the heap of a Grassmannian as follows:



Let $u = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2} \in W^{P_2}$ and v be any element in W^{P_2} with $l(v) = 5$. We have $\alpha = \alpha_2$ and $us_\alpha = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}$, thus its heap is not contained in the heap of w_Z and we get

$$\sigma^u * \sigma^v = \sigma^u \cup \sigma^v.$$

All the coefficients in this classical product can be computed using jeu de taquin.

Let us now set $u' = s_{\alpha_1}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2} \in W^{P_2}$ and $v' = s_{\alpha_1}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2} \in W^{P_2}$ as elements in W^{P_2}. We have $u's_\alpha = s_{\alpha_1}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}$ and $v's_\alpha = s_{\alpha_1}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}$; thus by jeu de taquin we get

$$\sigma^u * \sigma^v = \sigma^u \cup \sigma^v + q.$$

Again, all the coefficients in the classical product $\sigma^u \cup \sigma^v$ can be computed using jeu de taquin.

5. *Presentation of the quantum cohomology rings*

Recall that for the minuscule and cominuscule varieties, presentations of the quantum cohomology rings are already known: in type A this is done in [**30**], in type B, C and D in [**19**, **20**]. In [**9**] the authors together with L. Manivel gave a uniform presentation of the quantum cohomology rings of the minuscule and cominuscule homogeneous spaces. In this section we give a presentation of the quantum cohomology rings of adjoint and coadjoint homogeneous spaces. As already mentioned, for classical groups this is already known: in type A a presentation will follow from the quantum Chevalley formula (see Section 7), in type B, C and D, such a presentation is contained in the more general results of [**6**]. Therefore in this section we are concerned with exceptional types.

Before starting the case-by-case description, let us recall the definition of the quantum Hasse diagram. It is a graph whose vertices are indexed by the quantum monomials (that is, indexed by $W^P \times \mathbb{N}$) and such that there is an edge between two vertices indexed by $\sigma_u q^a$ and $\sigma_v q^b$ if one of the two monomials appear in the product of the other by the hyperplane class h (the number of edges corresponds to the multiplicity of the monomial in the product by h). We shall only draw parts of this diagram, in particular, only parts of the positive part of the diagram corresponding to monomials $\sigma_u q^d$ with $d \geqslant 0$.

Most of the computations here made use of a software written in Java that the reader may upload or test at the webpage www.math.sciences.univ-nantes.fr/∼chaput/quiver-demo.html, in order to visualize the quantum Hasse diagrams, apply our version of the Littlewood–Richardson rule [**12**], perform quantum multiplications and, in general, check the assertions in this section, see [**7**].

In the proofs in this section, the Hasse diagrams are colored: the doubled circled vertices correspond to ϖ-minuscule elements. The dark grey vertices correspond to quantum Schubert classes with a factor q while grey vertices correspond to quantum Schubert classes with a factor q^2. In the paper [**13**], we gathered the quantum Giambelli formulas for all exceptional adjoint and coadjoint varieties. We also give in [**13**] some tableaux used in the proof of the presentation for E_8/P_8 (we did not include these tableaux in this paper to save place).

In the Hasse diagrams displayed in this section, the vertex corresponding to the class h is always the second leftmost vertex. The vertex corresponding to q is always the leftmost dark grey vertex. We refer to [**14**] for diagrams with colors.

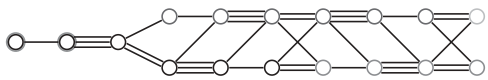
5.1. *Adjoint and coadjoint varieties in type* G_2

The homogeneous space G_2/P_1 is a smooth 5-dimensional quadric, thus its quantum cohomology ring is $QH^*(G_2/P_1, \mathbb{Q}) = \mathbb{Q}[h,q]/(4hq - h^6)$. For G_2/P_2 we have the following proposition.

PROPOSITION 5.1. *Let h denote the hyperplane class of G_2/P_2. We have*

$$QH^*(G_2/P_2) = \mathbb{Q}[h,q]/(h^6 - 18h^3q - 27q^2).$$

Proof. Using Theorem 3, it is immediate to check that the quantum Hasse diagram of G_2/P_2 begins as follows (and then it is periodic):



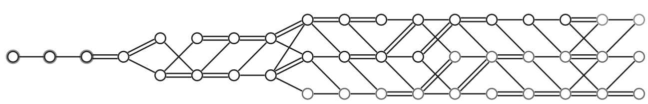
From this one computes that the degree 2 class is $\sigma_2 = 1/3h^2$ and that the class of a point is $\sigma_5 = h^5/18 - 5h^2q/6$. We deduce from this that $q^2 = (2h\sigma_5 - hq\sigma_2)/3 = h^6/27 - 2h^3q/3$. □

5.2. *Adjoint and coadjoint varieties in type* F_4

PROPOSITION 5.2. *Let h denote the hyperplane class of F_4/P_1 and $s \in H^*(F_4/P_1)$ be the Schubert class corresponding to $w = s_2s_3s_2s_1$. We have*

$$QH^*(F_4/P_1, \mathbb{Q}) = \mathbb{Q}[h,s,q]/(h^8 - 12s^2 - 16q, 3h^{12} - 18h^8s + 24h^4s^2 + 8s^3).$$

Proof. The vertex corresponding to the class of s, which is of degree 4, is the bottom vertex on the fifth column. The quantum Hasse diagram of F_4/P_1 begins as follows:

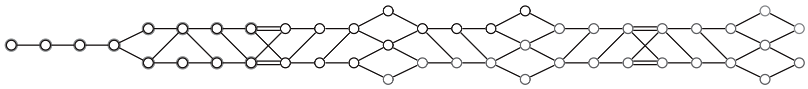


To compute s^2 in $QH^*(F_4/P_1)$, we first compute the classical product. Let $\sigma_{4,2}$ correspond to $s_4s_3s_2s_1$: the Hasse diagram shows that $\sigma_{4,2}^2$ has degree 56. Let $\sigma_{8,1}$ and $\sigma_{8,2}$ correspond to $s_1s_2s_3s_2s_4s_3s_2s_1$ and $s_2s_1s_3s_2s_4s_3s_2s_1$, respectively. Since $\deg(\sigma_{8,1}) = 16$ and $\deg(\sigma_{8,2}) = 40$, we deduce that $\sigma_{4,2} \cup \sigma_{4,2} = \sigma_{8,1} + \sigma_{8,2}$. Using the Chevalley formula, it follows that $s \cup s = 6\sigma_{8,1} + 8\sigma_{8,2}$. The quantum product $s * s = s^2$ is then equal to the same class by Example 4.4. By the quantum Chevalley formula, we get the first relation. The second relation (of degree 12) is then just a consequence of the Chevalley formula. This is enough in order to express all the Schubert classes as polynomials in h, s and q, as displayed in [**13**]. □

PROPOSITION 5.3. *Let h denote the hyperplane class of F_4/P_4 and $s \in H^*(F_4/P_1)$ be the Schubert class corresponding to $w = s_1s_2s_3s_4$. We have*

$$QH^*(F_4/P_4, \mathbb{Q}) = \mathbb{Q}[h,s,q]/(2h^8 - 6h^4s + 3\ s^2, -11h^{12} + 26h^8s + 3hq).$$

Proof. The vertex corresponding to the class of s, which is of degree 4, is the top vertex on the fifth column. The quantum Hasse diagram of F_4/P_4 begins as follows:



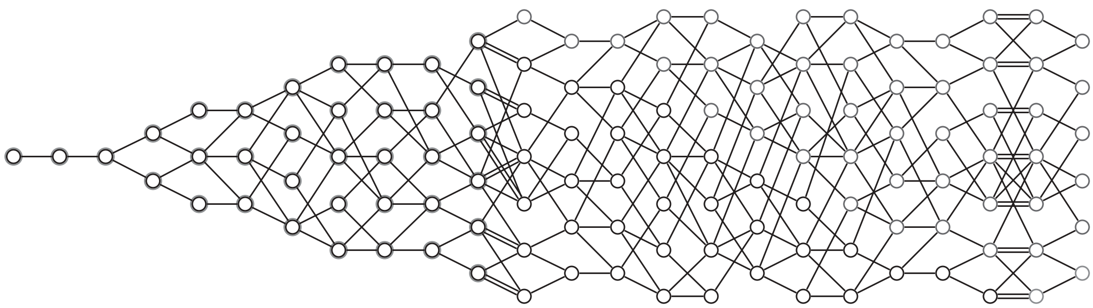
This Hasse diagram shows that the classical (or quantum because its cohomological degree is $8 < c_1$) product $s \cup s = s * s$ has degree 14. Let us denote by $\sigma_{8,1}$ and $\sigma_{8,2}$ the Schubert classes corresponding to $s_3s_2s_4s_1s_3s_2s_3s_4$ resp. $s_4s_3s_2s_1s_3s_2s_3s_4$ of degrees 5 and 2, respectively. We have either $s^2 = 2\sigma_{8,1} + 2\sigma_{8,2}$ or $s^2 = 7\sigma_{8,2}$. Since $s * h^4 \geqslant s^2$ and $s * h^4 = 5\sigma_{8,1} + 4\sigma_{8,2}$, the second case is excluded. Thus we have $s^2 = 2\sigma_{8,1} + 2\sigma_{8,2}$. The relations in degree 8 and 12 follow. □

5.3. *Adjoint and coadjoint varieties in type E_6*

We now consider the technically more involved cases of groups of type E. In the cohomology of E_6/P_2, let h denote the hyperplane class, and s and t the classes corresponding to the elements $s_3s_4s_2$ and $s_1s_3s_4s_2$, respectively.

PROPOSITION 5.4. *The quantum cohomology algebra $QH^*(E_6/P_2, \mathbb{Q})$ is the quotient of the polynomial algebra $\mathbb{Q}[h, s, t, q]$ by the relations $h^8 - 6h^5s + 3h^4t + 9h^2s^2 - 12hst + 6t^2$, $h^9 - 4h^6s + 3h^5t + 3h^3s^2 - 6h^2st + 2s^3$ and $-97h^{12} + 442h^9s - 247h^8t - 507h^6s^2 + 624h^5st - 156h^2s^2t + 48hq$.*

Proof. The vertices corresponding to the classes of s and t, which are of degree 3 and 4, are the top vertices on the fourth and fifth columns, respectively. We know that $QH^*(E_6/P_2, \mathbb{Q})$ is generated by h, s and t and that there are relations in degrees 8, 9 and 12. The quantum Hasse diagram of E_6/P_2 begins as follows:



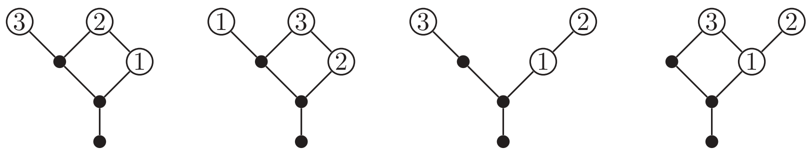
From this it follows that the degree of s^2 is 37752. From now on and until the end of this section, we write Schubert classes according to the following notation in the proofs.

NOTATION 5.5. To $w \in W/W_P$ we associate the element $\varphi(w) = \alpha = \varpi - w(\varpi)$ of the root lattice. If $\alpha = \varphi(w)$ with $w \in W/W_P$, then we denote by $\sigma(\alpha)$ the cohomology class σ_w.

Let us define the following Schubert classes

$$\sigma_{6,1} = \sigma\begin{pmatrix} 1\,1\,2\,1\,0 \\ 2 \end{pmatrix}, \quad \sigma_{6,2} = \sigma\begin{pmatrix} 0\,1\,2\,1\,0 \\ 2 \end{pmatrix}, \quad \sigma_{6,3} = \sigma\begin{pmatrix} 1\,1\,1\,1\,1 \\ 1 \end{pmatrix} \quad \text{and} \quad \sigma_{6,4} = \sigma\begin{pmatrix} 0\,1\,2\,1\,1 \\ 1 \end{pmatrix},$$

they have respective degrees 10920, 6006, 4992 and 10920. Considering the following tableaux, we deduce from [12] that $s^2 \geqslant 2\sigma_{6,1} + \sigma_{6,3} + \sigma_{6,4}$ (the Littlewood–Richardson rule proved in [12] says that the coefficient in the product σ^2 is equal to the number of tableaux rectifying to the given class therefore producing tableaux will always give inequalities of this type):

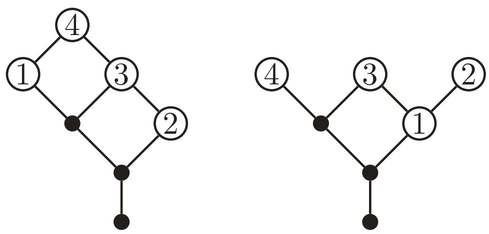


On the other hand both s^2 and $2\sigma_{6,1} + \sigma_{6,3} + \sigma_{6,4}$ have degree 37752; thus we have $s^2 = 2\sigma_{6,1} + \sigma_{6,3} + \sigma_{6,4}$. This allows to compute an expression of all Schubert classes up to degree 6 as polynomials in h, s, t, as displayed in [13].

Let $\sigma_{7,i}, 1 \leqslant i \leqslant 5$, be the Schubert classes in $H^7(E_6/P_2)$, corresponding, respectively, to the roots

$$\begin{pmatrix}1\,1\,2\,1\,0\\ 2\end{pmatrix}, \quad \begin{pmatrix}1\,2\,2\,1\,0\\ 1\end{pmatrix}, \quad \begin{pmatrix}1\,1\,2\,1\,1\\ 1\end{pmatrix}, \quad \begin{pmatrix}0\,1\,2\,1\,1\\ 2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\,1\,2\,2\,1\\ 1\end{pmatrix}.$$

They have the respective degrees 3003, 2925, 4992, 3003 and 2925. The product st has degree 7917. In view of the two tableaux

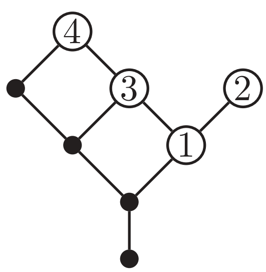


it follows that $st = \sigma_{7,2} + \sigma_{7,3}$.

This is enough to express some Giambelli formulas up to degree 9 but to get the relation in degrees 8 and 9, we must compute t^2 and s^3. To this end we compute t^2 and $s \cdot \sigma_{6,1}$. The degree of t^2 and

$$\sigma\begin{pmatrix}1\,2\,2\,1\,1\\ 1\end{pmatrix}$$

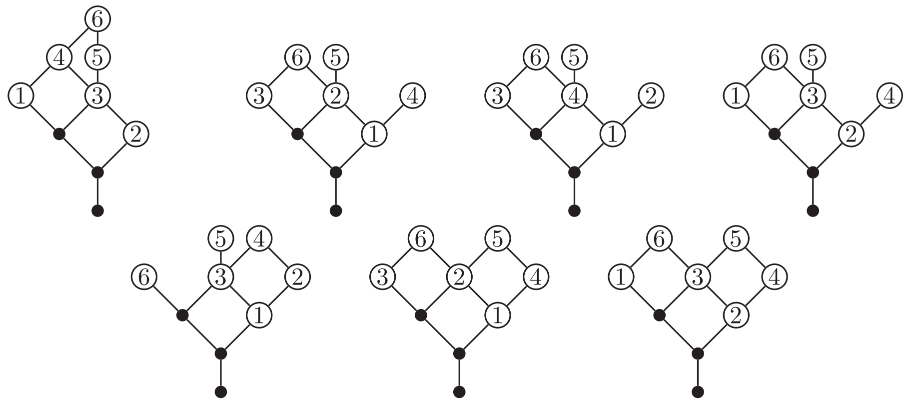
are 1638, and we have the following tableau:



thus

$$t^2 = \sigma\begin{pmatrix}1\,2\,2\,1\,1\\ 1\end{pmatrix}.$$

The following seven tableaux:

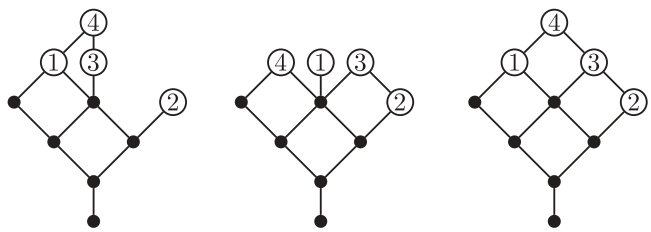


prove the inequality

$$s \cdot \sigma_{6,1} \geqslant \sigma\begin{pmatrix}1\,2\,3\,1\,0\\2\end{pmatrix} + 3\sigma\begin{pmatrix}1\,2\,2\,1\,1\\2\end{pmatrix} + \sigma\begin{pmatrix}1\,1\,2\,2\,1\\1\end{pmatrix} + 2\sigma\begin{pmatrix}1\,2\,2\,2\,1\\1\end{pmatrix}.$$

We leave it to the reader to check that the degrees coincide, so that we have equality.

To express the Schubert classes in $H^{10}(E_6/P_2)$ as polynomials in h, s, t, we must compute s^2t, and to this end we compute the product $t\sigma_{6,1}$. According to the following three tableaux:



and using the same degree argument as above this product is

$$\sigma\begin{pmatrix}1\,2\,3\,1\,1\\2\end{pmatrix} + \sigma\begin{pmatrix}1\,2\,2\,2\,1\\2\end{pmatrix} + \begin{pmatrix}1\,2\,3\,2\,1\\1\end{pmatrix}.$$

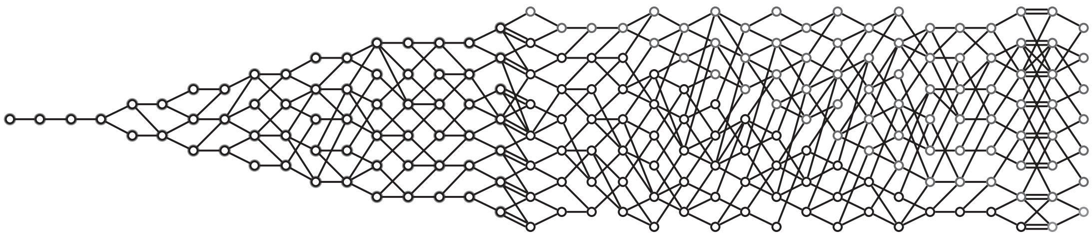
These computations are enough to express all the Schubert classes as polynomials in h, s, t, q, and the displayed relation in degree 12 comes for free. □

5.4. *Adjoint and coadjoint varieties in type E_7*

In the cohomology of E_7/P_1, let h denote the hyperplane class, and s and t the classes corresponding to the elements $s_2s_4s_3s_1$ and $s_7s_6s_5s_4s_3s_1$, respectively.

Proposition 5.6. *The quantum cohomology algebra $QH^*(E_7/P_1, \mathbb{Q})$ is the quotient of the polynomial algebra $\mathbb{Q}[h, s, t, q]$ by the relations $h^{12} - 6h^8s - 4h^6t + 9h^4s^2 + 12h^2st - s^3 + 3t^2$, $h^{14} - 6h^{10}s - 2h^8t + 9h^6s^2 + 6h^4st - h^2s^3 + 3s^2t$ and $232h^{18} - 1444h^{14}s - 456h^{12}t + 2508h^{10}s^2 + 1520h^8st - 988h^6s^3 + 133h^2s^4 + 36hq$.*

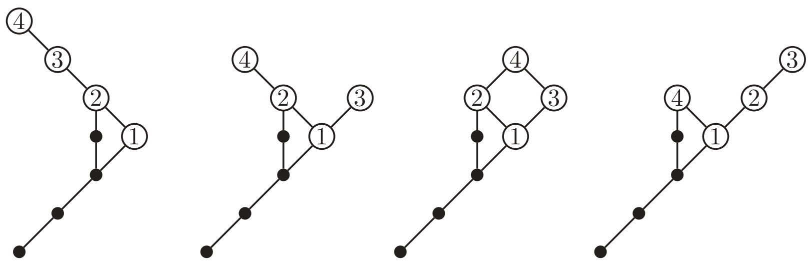
Proof. The vertices corresponding to the classes of s and t, which are of degree 4 and 6, are the bottom and top vertices on the fifth and seventh columns, respectively. We know that $QH^*(E_7/P_1, \mathbb{Q})$ is generated by h, s and t and that there are relations in degrees 12, 14 and 18. The quantum Hasse diagram of E_7/P_1 begins as follows:



The strategy of proof is the same as in the previous cases; therefore we will only sketch the arguments. We can express the Giambelli formulas up to degree 7 without computing any product. In degree 8 we need to compute s^2. We have

$$s^2 = \sigma\begin{pmatrix}2\,2\,2\,1\,0\,0\\1\end{pmatrix} + \sigma\begin{pmatrix}1\,2\,2\,1\,1\,0\\1\end{pmatrix} + \sigma\begin{pmatrix}1\,1\,2\,2\,1\,0\\1\end{pmatrix} + \sigma\begin{pmatrix}1\,1\,2\,1\,1\,1\\1\end{pmatrix}.$$

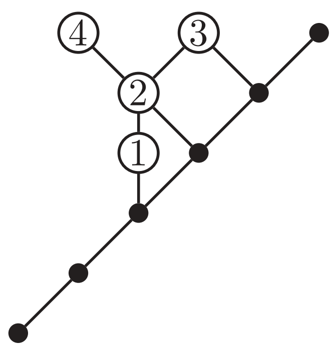
In fact, computing degrees, this follows from the existence of the following four tableaux:



In degree 10, we have to compute st: this is equal to

$$\sigma\begin{pmatrix}1\,2\,2\,2\,1\,1\\1\end{pmatrix},$$

in view of



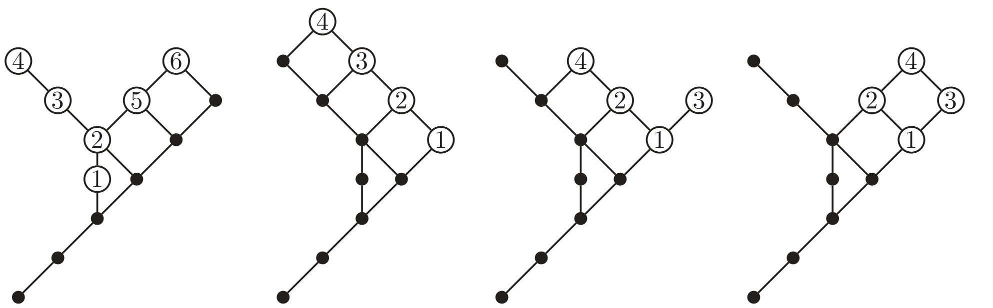
In degree 12, we have two new monomials: t^2 and s^3. We have

$$t^2 = \sigma\begin{pmatrix}222221\\1\end{pmatrix}.$$

To express s^3 as a sum of Schubert classes and thus find the given relation of degree 12, we compute that

$$s \cdot \sigma\begin{pmatrix}222100\\1\end{pmatrix} = \sigma\begin{pmatrix}233210\\1\end{pmatrix} + \sigma\begin{pmatrix}223211\\1\end{pmatrix} + \sigma\begin{pmatrix}222221\\1\end{pmatrix}$$

The corresponding tableau for t^2 is displayed on the left and the three tableaux for the above product are on the right:



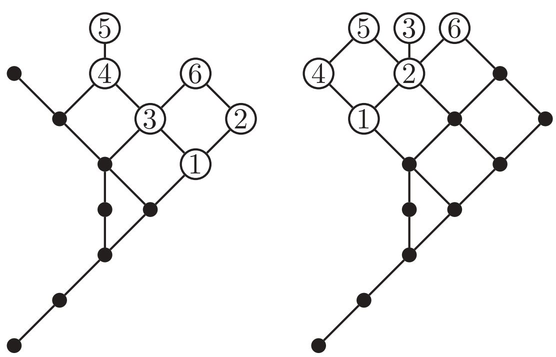
To get the relation in degree 14, we have to compute the monomial s^2t. To this end we compute that

$$t \cdot \sigma\begin{pmatrix}222100\\1\end{pmatrix} = \sigma\begin{pmatrix}223221\\2\end{pmatrix}.$$

To express the Schubert classes in $H^{16}(E_7/P_1)$, we need to compute st^2, and to this end we show that

$$t \cdot \sigma\begin{pmatrix}112221\\1\end{pmatrix} = \sigma\begin{pmatrix}233321\\2\end{pmatrix}.$$

These relations come from the two tableaux



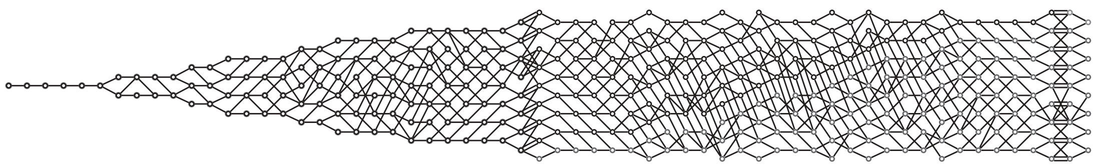
With this we express all Schubert classes as polynomials in h, s, t, q, and get the last relation. $\square$

5.5. *Adjoint and coadjoint varieties in type E_8*

In the cohomology of E_8/P_8, let h denote the hyperplane class, and s and t the classes corresponding to the elements $s_2s_4s_5s_6s_7s_8$ and $s_6s_5s_4s_3s_2s_4s_5s_6s_7s_8$, respectively.

PROPOSITION 5.7. *The quantum cohomology algebra $QH^*(E_8/P_8,\mathbb{Q})$ is the quotient of the polynomial algebra $\mathbb{Q}[h,s,t,q]$ by the relations* $h^{14}s+6h^{10}t-3h^8s^2-12h^4st-10h^2s^3+3t^2$, $29h^{24}-120h^{18}s+15h^{14}t+45h^{12}s^2-30h^8st+180h^6s^3-30h^2s^2t+5s^4$ *and* $-86357h^{30}+368652h^{24}s-44640h^{20}t-189720h^{18}s^2+94860h^{14}st-473680h^{12}s^3+74400h^8s^2t-1240h^2s^3t+60hq$.

Proof. The vertices corresponding to the classes of s and t, which are of degree 6 and 10, are the top and bottom vertices on the seventh and eleventh columns, respectively. We know that $QH^*(E_8/P_8,\mathbb{Q})$ is generated by h,s and t and that there are relations in degrees 14, 24 and 30. The quantum Hasse diagram of E_8/P_8 begins as follows:



From this we may compute the degrees of all the Schubert classes and of any product of two Schubert classes. Moreover, we argue as before to compute such products, and display the results in the companion paper [**13**]. We only indicate the computed products, and why they suffice to obtain the presentation. Note that these results were obtained with the help of a computer by Nikolenko and Semenov [**24**]. We have

$$s^2=2\sigma\binom{1\,2\,3\,2\,1\,1\,1}{1}+4\sigma\binom{1\,2\,2\,2\,2\,1\,1}{1}+2\sigma\binom{1\,1\,2\,2\,2\,2\,1}{1}+\sigma\binom{0\,1\,2\,2\,2\,2\,2}{1}$$

and

$$s\cdot t=\sigma\binom{1\,2\,4\,3\,2\,1\,1}{2}+5\sigma\binom{1\,2\,3\,3\,2\,2\,1}{2}+6\sigma\binom{1\,2\,3\,2\,2\,2\,2}{2}+2\sigma\binom{1\,2\,3\,3\,3\,2\,1}{1}+7\sigma\binom{1\,2\,3\,3\,2\,2\,2}{1}.$$

This allows to compute some Giambelli formulas up to degree 17. To compute s^3 we compute

$$s\cdot\sigma\binom{0\,1\,2\,2\,2\,2\,2}{1}=\sigma\binom{1\,2\,4\,3\,2\,2\,2}{2}+\sigma\binom{1\,2\,3\,3\,3\,2\,2}{2}+\sigma\binom{1\,2\,3\,3\,3\,3\,2}{1},$$

which allows to get the Giambelli formulas up to degree 21. To get the relation in degree 20 we compute

$$\begin{aligned}t^2 \;=\; & 4\sigma\binom{2\,3\,4\,3\,3\,2\,1}{2}+2\sigma\binom{1\,3\,4\,4\,3\,2\,1}{2}+7\sigma\binom{2\,3\,4\,3\,2\,2\,2}{2}+14\sigma\binom{1\,3\,4\,3\,3\,2\,2}{2}\\ & +8\sigma\binom{1\,2\,4\,4\,3\,2\,2}{2}+16\sigma\binom{1\,2\,4\,3\,3\,3\,2}{2}.\end{aligned}$$

To express the Schubert classes of degree 22, we need to compute s^2t: to this end we compute

$$t\cdot\sigma\binom{0\,1\,2\,2\,2\,2\,2}{1}=\sigma\binom{2\,3\,4\,4\,3\,2\,2}{2}+3\sigma\binom{1\,3\,5\,4\,3\,3\,2}{3}+2\sigma\binom{2\,3\,4\,4\,4\,3\,2}{2}+\sigma\binom{1\,2\,4\,3\,3\,3\,2}{2}.$$

To get the relation in degree 24, we need to know s^4, and for this we show

$$\sigma\begin{pmatrix}0\,1\,2\,2\,2\,2\,2\\1\end{pmatrix}^2 = \sigma\begin{pmatrix}2\,3\,4\,4\,4\,3\,2\\2\end{pmatrix}.$$

Finally we need to express the Schubert classes of degree 28 in terms of the generators; for this we need to compute s^3t, which follows from the equality

$$\sigma\begin{pmatrix}0\,1\,2\,2\,2\,2\,2\\1\end{pmatrix}\cdot\sigma\begin{pmatrix}1\,2\,3\,3\,2\,2\,2\\1\end{pmatrix} = \sigma\begin{pmatrix}2\,4\,6\,4\,4\,3\,2\\3\end{pmatrix}+\sigma\begin{pmatrix}2\,4\,6\,5\,4\,3\,2\\2\end{pmatrix}$$
$$+4\sigma\begin{pmatrix}2\,4\,5\,5\,4\,3\,2\\3\end{pmatrix}+\sigma\begin{pmatrix}2\,3\,6\,5\,4\,3\,2\\3\end{pmatrix}.$$

The relation in degree 30 follows from the Chevalley formula. □

6. *Semi-simplicity of the quantum cohomology and strange duality*

Recall that, in [8, 11], we proved that the localization

$$QH^*(X,\mathbb{C})_{\mathrm{loc}} := QH^*(X,\mathbb{C})[q^{-1}]$$

of the quantum parameter in the quantum cohomology algebra of any minuscule or cominuscule variety X is semi-simple. A general argument using complex conjugation (see [11, Theorem 2.1]) implies that in that case there exists an algebra involution on $QH^*(X,\mathbb{C})_{\mathrm{loc}}$ sending q to its inverse and more generally a quantum cohomology class of degree d to a quantum cohomology class of degree $-d$. We conjectured in [11] that the only rational homogeneous spaces X with Picard number 1, such that $QH^*(X,\mathbb{C})_{\mathrm{loc}}$ is semi-simple, were the minuscule and cominuscule ones.

6.1. *Coadjoint varieties*

THEOREM 5. *Assume that there exists a presentation of the classical cohomology algebra $H^*(X,\mathbb{C})$ satisfying the following conditions*:
(i) *there is no equation of degree a multiple of $c_1(X)$;*
(ii) *there is at least one equation of degree smaller than $c_1(X)$.*
Then the localized quantum cohomology algebra $QH^*(X,\mathbb{C})$ is not semi-simple.

Proof. First remark that, to prove that the localized quantum cohomology algebra $QH^*(X,\mathbb{C})_{\mathrm{loc}}$ is not semi-simple, it suffices to prove that the specialization of the quantum cohomology algebra at $q=1$ is not semi-simple.

Let us fix some notation for the presentation of the cohomology ring $H^*(X,\mathbb{C})$. This ring can always be presented as a homogeneous complete intersection ring: the generators and the equations are homogeneous and there are as many equations as there are generators. Let us denote by $(X_i)_{i\in[1,s]}$ the generators and by $(F_i)_{i\in[1,s]}$ the equations. The generators will be called classical generators and the equations classical equations.

A presentation of the quantum cohomology algebra can be derived from a presentation of the classical cohomology algebra by taking the same classical generators together with q as generators and by deforming the classical relations F_i in relations G_i of the same degree but possibly with q terms (see [30]). By hypothesis we get homogeneous equations of degree different from multiples of c_1. In particular any monomial with a q term is also a multiple of a classical generator. This implies that $X_i=0$ for all $i\in[1,s]$ and $q=1$ is a solution of the equations $((G_i=0)_{i\in[1,s]}, q=1)$. Denote by 0 this point. We want to prove that 0 is a multiple solution of the system $((G_i=0)_{i\in[1,s]}, q=1)$.

For this, because the specialization at $q = 1$ of the quantum algebra $QH^*(X, \mathbb{C})$ is a complete intersection algebra of dimension 0, we only need to prove that the family $(d_0G_i)_{i\in[1,s]}$ of differentials is not linearly independent. But an equation F_i is not deformed if $\deg(F_i) < \deg(q) = c_1(X)$. In that case, $G_i = F_i$ and by minimality F_i is at least quadratic in the variables $(X_i)_{i\in[1,s]}$, therefore its differential d_0F_i vanishes. In particular the family $(d_0G_i)_{i\in[1,s]}$ of differentials is not linearly independent and 0 is a multiple solution. □

COROLLARY 6.1. *For X a coadjoint variety for a semi-simple group different from G_2, the localized quantum cohomology algebra $QH^*(X, \mathbb{C})_{\mathrm{loc}}$ is not semi-simple.*

6.2. *Adjoint non-coadjoint varieties*

There are three adjoint non-coadjoint varieties: $\mathbb{G}_Q(2, 2n+1)$, F_4/P_1 and G_2/P_2. In this subsection we prove the following.

THEOREM 6. *For X an adjoint non-coadjoint rational homogeneous space, the localized quantum cohomology algebra $QH^*(X, \mathbb{Q})_{\mathrm{loc}}$ is semi-simple.*

We proved with Manivel in [**11**, Theorem 2.1] that, for any smooth projective variety X, if the quantum cohomology $QH^*(X, \mathbb{Z})$ specialized at $q = 1$ is semi-simple, then its localization $QH^*(X, \mathbb{Z})_{\mathrm{loc}}$ admits an algebra involution ι sending q to its inverse. We therefore get involutions on the quantum cohomology of any adjoint non-coadjoint homogeneous space.

We prove this result by a case-by-case analysis. We start with type B_n.

PROPOSITION 6.2. *The localized quantum cohomology algebra $QH^*(\mathbb{G}_Q(2, 2n+1), \mathbb{Q})_{\mathrm{loc}}$ is semi-simple.*

Proof. A presentation of the ring $QH^*(\mathbb{G}_Q(2, 2n+1), \mathbb{Z})$ (over the integers) has been given in [**6**]. We shall use a simplified presentation using a parametrization of the Chern classes of the tautological bundles by their Chern roots. The presentation is only valid over $\mathbb{Q}$, but this is enough for our purpose.

Let us denote by K and Q the tautological subbundle and quotient bundle on $\mathbb{G}_Q(2, 2n+1)$, respectively. The rank of K is 2 while the rank of Q is $2n-1$. By the splitting principle, we may formally work with Chern roots. Let us thus denote by x_1, x_2 the Chern roots of K. Because the quotient $Q^\vee/K$ is self-dual, there are Chern roots $x_3, \ldots, x_n$ such that the following equalities holds in $H^* := H^*(\mathbb{G}_Q(2, 2n+1), \mathbb{Q})$:

$$c_t(K) = (1 + x_1t)(1 + x_2t) \quad \text{and} \quad c_t(Q) = (1 - x_1t)(1 - x_2t)\prod_{i=3}^{n}(1 - x_i^2t^2),$$

where c_t denotes the Chern polynomial. The tautological exact sequence gives $c_t(K)c_t(Q) = 1$ in H^*. We can compute explicit formulas for the Chern classes of K and Q. Let us write $c_i = c_i(K)$ for $i \in [0, 2]$ and $\psi_j = c_j(Q)$ for $j \in [0, 2n-2]$. We have the following equalities in H^*.

$$\begin{aligned} c_0 = \psi_0 = 1, \quad c_1 = x_1 + x_2, \quad c_2 = x_1x_2,& \\ \psi_{2j} = (-1)^j\sigma_j(x_3^2, \ldots, x_n^2) + (-1)^{j-1}x_1x_2\sigma_{j-1}(x_3^2, \ldots, x_n^2),& \\ \psi_{2j+1} = (-1)^{j+1}(x_1 + x_2)\sigma_j(x_3^2, \ldots, x_n^2),& \end{aligned} \tag{3}$$

where $\sigma_j(x_3^2,\dots,x_n^2)$ is the jth elementary symmetric function in the $n-2$ variables $x_3^2,\dots,x_n^2$. Remark that $\sigma_{n-1}(x_3^2,\dots,x_n^2)=0$; thus we have

$$\psi_{2n-2}=(-1)^n\sigma_{n-2}(x_3^2,\dots,x_n^2)\quad\text{and}\quad\psi_{2n-1}=0. \tag{4}$$

The relation $c_t(K)c_t(Q)=1$ yields a presentation of the cohomology algebra:

$$H^*(\mathbb{G}_Q(2,2n+1),\mathbb{Q})=\mathbb{Q}[c_1,c_2,(\psi_j)_{j\in[1,2n-2]}]/(\Sigma_k(x_l^2)_{k\in[1,n]}), \tag{5}$$

where $\Sigma_j(x_1^2,\dots,x_n^2)$ is the jth elementary symmetric function in the n variables $x_1^2,\dots,x_n^2$. This is also the Borel presentation (see [**2**] or [**1**]). Using the first $n-2$ relations, we can express the Chern classes $(\psi_j)_{j\in[1,2n-2]}$ in terms of c_1 and c_2, but we keep expressions in x_1 and x_2. We get the following formulas in H^*:

$$\sigma_j(x_3^2,\dots,x_n^2)=(-1)^j\sum_{k=0}^{j}x_1^{2k}x_2^{2(j-k)}\quad\text{for } j\in[1,n-2],$$
$$\psi_j=(-1)^j\sum_{k=0}^{j}x_1^{k}x_2^{j-k}\quad\text{for } j\in[1,2n-3].$$

Remark that because these equalities involve only elements of degree smaller that $2n-2$, the degree of the quantum parameter, they are also valid in $QH^*:=QH^*(G_Q(2,2n+1),\mathbb{Q})$.

Using the formulas (3), we get the following equations in H^*:

$$\psi_{2n-2}+c_1\psi_{2n-3}+c_2\psi_{2n-4}=(-1)^{n-1}\Sigma_{n-1}(x_l^2)=0\quad\text{and}\quad c_2\psi_{2n-2}=(-1)^{n-2}\Sigma_n(x_l^2)=0. \tag{6}$$

In these expressions, the ψ terms can be removed but we shall keep them for the moment as they are easier to express in terms of Schubert varieties. This will be useful to deform these equations in the quantum cohomology ring. Let us describe the classes ψ_{2n-4}, ψ_{2n-3} and ψ_{2n-2} as Schubert classes. This is classical (see, for example, [**6**]). Let $(W_p)_{p\in[1,n]}$ be a complete isotropic flag; then we have the following equalities

$$\psi_{2n-4}=2[\{V_2\in\mathbb{G}_Q(2,2n+1)\mid\dim(W_3\cap V_2)\geqslant 1\}],$$
$$\psi_{2n-3}=2[\{V_2\in\mathbb{G}_Q(2,2n+1)\mid\dim(W_2\cap V_2)\geqslant 1\}],$$
$$\psi_{2n-2}=2[\{V_2\in\mathbb{G}_Q(2,2n+1)\mid\dim(W_1\cap V_2)\geqslant 1\}],$$

where $[Z]$ denotes the cohomology class of Z. In terms of roots as described in Section 2 (with notation as in [**3**]), we have $\psi_{2n-4}=2\sigma_{\alpha_1+\alpha_2}$, $\psi_{2n-3}=2\sigma_{\alpha_1}$ and $\psi_{2n-2}=2\sigma_{-\alpha_1}$. The Chern classes c_1 and c_2 are also related to Schubert classes:

$$c_1=[\{V_2\in\mathbb{G}_Q(2,2n+1)\mid\dim(W_2^\perp\cap V_2)\geqslant 1\}],$$
$$c_2=[\{V_2\in\mathbb{G}_Q(2,2n+1)\mid V_2\subset W_1^\perp\}]$$

or with the notation of Section 2: $c_1=\sigma_{\Theta-\alpha_2}$ and $c_2=\sigma_{\Theta-\alpha_2-\alpha_1}$.

To compute quantum cohomology, we need to deform the preceding relations using the quantum parameter q. Remark that the degree of q is $2n-2$; therefore only the last two equations in the presentation (5) can be deformed. We thus need to compute the quantum terms in the equations (6). Now using Proposition 4.5, we can compute that $c_2*\sigma_{\alpha_1+\alpha_2}=c_2\cup\sigma_{\alpha_1+\alpha_2}$ and that $c_1*\sigma_{\alpha_1}=c_1\cup\sigma_{\alpha_1}+q$. We get

$$\psi_{2n-2}+c_1*\psi_{2n-3}+c_2*\psi_{2n-4}=2q. \tag{7}$$

In particular, replacing the values of c_i and ψ_j for $j \leqslant 2n-3$ in terms of x_1 and x_2, we get

$$\psi_{2n-2} = \sum_{k=0}^{2n-2} x_1^k x_2^{2n-2-k} + 2q$$

(in this expression the symmetric polynomial in x_1, x_2 is interpreted as a polynomial in c_1 and c_2, itself computed in the quantum ring). To deform the last equation, we only need to use the affine symmetries (see Subsection 2.5). Indeed $\frac{1}{2}\psi_{2n-2}$ corresponds to the opposite of the cominuscule simple root, and, for τ_c defined there, we have $\tau_c(\alpha_1) = -\Theta$ and $\tau_c(\alpha_2) = \alpha_2$. We get

$$c_2 * \psi_{2n-2} = 2qc_2. \tag{8}$$

Because we used one equation to express ψ_{2n-2} in terms of the Chern roots x_1 and x_2, we need one more equation given in [**6**]:

$$\psi_{n-1}^2 + 2\sum_{k=1}^{n-1}(-1)^k \psi_{n-1-k}\psi_{n-1+k} = 0. \tag{9}$$

Expressing the Chern classes $(\psi_j)_{j\in[1,2n-2]}$ in terms of the Chern roots x_1 and x_2, we get the equality in QH^*:

$$\sum_{k=0}^{2n-2} x_1^{2k} x_2^{2n-2-2k} + 4q = 0.$$

To show the semi-simplicity of the quantum cohomology ring, we only need to prove that when we set $q = 1$, the scheme $\mathrm{Spec}(QH^*(\mathbb{G}_Q(2, 2n+1), \mathbb{Q})_{q=1})$ is reduced. In turn, solving equations (8) and (9) we get that $QH^*(\mathbb{G}_Q(2, 2n+1), \mathbb{Q})$ is the ring of S_2-invariants of the quotient of $\mathbb{Q}[x_1, x_2]$ by the equations

$$\sum_{k=0}^{2n-2} x_1^{2k} x_2^{2n-2-2k} + 4q = 0 \quad \text{and} \quad x_1x_2\left(\sum_{k=0}^{2n-2} x_1^k x_2^{2n-2-k} + 2q\right) = 2qx_1x_2.$$

It is enough to check that this quotient is reduced.

To solve these equations for $q = 1$, we first remove the solutions (x_1, x_2) with $x_i = 0$ and x_{3-i} a $2(n-1)$th root of -4. There are $4(n-1)$ distinct such solutions. Then we assume $x_1x_2 \neq 0$; we get

$$\sum_{k=0}^{2n-2} x_1^{2k} x_2^{2n-2-2k} + 4 = 0 \quad \text{and} \quad \sum_{k=0}^{2n-2} x_1^k x_2^{2n-2-k} = 0.$$

The second equation gives $x_2 = \zeta x_1$ with $\zeta^{2n-1} = 1$ and $\zeta \neq 1$. Replacing in the first one we get $x_1^{2n-2} = -4(\zeta + 1)$ and there are $2n-2$ solutions for ζ and $2n-2$ solutions for x_1 when ζ is fixed. We described $4(n-1) + (2n-2)(2n-2) = 2n(2n-2)$ solutions. Since this number is equal to the dimension of $QH^*(\mathbb{G}_Q(2, 2n+1))_{q=1}$, the above affine scheme is reduced. □

PROPOSITION 6.3. *The localized quantum cohomology algebra $QH^*(F_4/P_1, \mathbb{Q})_{\mathrm{loc}}$ is semi-simple.*

Proof. Recall that we have a presentation of the quantum cohomology ring given by

$$QH^*(F_4/P_1, \mathbb{C}) = \mathbb{C}[h, s, q]/(-h^8 + 12s^2 + 16q, 3h^{12} - 18h^8 s + 24h^4 s^2 + 8s^3),$$

where h is of degree 1 and s of degree 4. Recall also that $c_1 = 8$. We solve these equations in h and s when setting $q = 1$. First remark that there is no solution on the closed subset $s = 0$ nor on the closed subset $h = 0$. Eliminating h^8 in the second equation using the first one, we get

$$3h^4(5s^2 + 4) = 4s(13s^2 + 18).$$

Taking the square of this equation and eliminating h^8 again, we get

$$s^6 - 108s^4 - 576s^2 - 576 = 0. \tag{10}$$

For $P(S) = S^3 - 108S^2 - 576S - 576$ we have $P(-2) = 136$ and $P(0) = -576$, thus (10) is a degree 3 equation in s^2 whose solutions are real and distinct. The result follows. □

PROPOSITION 6.4. *The localized quantum cohomology algebra $QH^*(G_2/P_2, \mathbb{Q})_{\mathrm{loc}}$ is semi-simple.*

Proof. Recall that we have a presentation of the quantum cohomology ring given by

$$QH^*(G_2/P_2, \mathbb{C}) = \mathbb{C}[h, q]/(h^6 - 18h^3q - 27q^2)$$

where h is of degree 1. Recall also that $c_1 = 3$. We solve this equation in h when setting $q = 1$. This is a degree 2 equation in h^3 whose solutions are real non-vanishing and distinct. □

To conclude this subsection, we display in the following array the known results on the semi-simplicity of the localization of the quantum cohomology of homogeneous spaces with Picard number. Recall that we proved, in [**11**], that for X minuscule or cominuscule the localized quantum cohomology ring $QH^*(X, \mathbb{C})_{\mathrm{loc}}$ is semi-simple.

Type	Variety X	Condition	Semi-simplicity of $QH^*(X, \mathbb{C})_{\mathrm{loc}}$
A_n	$\mathbb{G}(p, n+1)$	$p \in [1, n]$	yes
B_n	$\mathbb{G}_Q(p, 2n+1)$	$p = 1$ or $p = n$	yes
B_n	$\mathbb{G}_Q(p, 2n+1)$	$p \in [2, n-1]$ even	?
B_n	$\mathbb{G}_Q(p, 2n+1)$	$p \in [2, n-1]$ odd	no
C_n	$\mathbb{G}_\omega(p, 2n)$	$p = 1$ or $p = n$	yes
C_n	$\mathbb{G}_\omega(p, 2n)$	$p \in [2, n-1]$ even	no
C_n	$\mathbb{G}_\omega(p, 2n)$	$p \in [2, n-1]$ odd	?
D_n	$\mathbb{G}_Q(p, 2n)$	$p = 1$ or $p = n$	yes
D_n	$\mathbb{G}_Q(p, 2n)$	$p \in [2, n-2]$ even	no
D_n	$\mathbb{G}_Q(p, 2n)$	$p \in [2, n-2]$ odd	?
E_6	$E_6/P_1, E_6/P_6$		yes
E_6	$E_6/P_2, E_6/P_4$		no
E_6	$E_6/P_3, E_6/P_5$		?
E_7	E_7/P_7		yes
E_7	$E_7/P_2, E_7/P_4, E_7/P_5$		?
E_7	$E_7/P_1, E_7/P_3, E_7/P_6$		no
E_8	E_8/P_6		?
E_8	E_8/P_i	$i \in \{1, 2, 3, 4, 5, 7, 8\}$	no
F_4	F_4/P_1		yes
F_4	F_4/P_2		?
F_4	$F_4/P_3, F_4/P_4$		no
G_2	$G_2/P_1, G_2/P_2$		yes

6.3. *Some properties of the involution for adjoint non-coadjoint varieties*

Recall that in the minuscule and cominuscule cases, it was proved in [**8**, **11**] that the involution ι given by complex conjugation has the nice property of sending a Schubert class to a multiple of such a class. In this section we shall see that, contrary to these cases, this property is not satisfied for adjoint non-coadjoint varieties.

We shall first prove a nice property of the involution ι, namely, it sends a class to a multiple of itself and the quantum parameter q for classes of degree c_1.

PROPOSITION 6.5. *For X an adjoint non-coadjoint rational homogeneous space, let σ be a quantum Schubert monomial of degree dc_1 for some integer d; then we have*

$$\iota(\sigma) = \frac{\sigma}{q^{2d}}.$$

Proof. We only give here a sketch of proof and refer to [**14**, Proposition 6.7] for the details of the proof. To prove this proposition, it is enough to specialize q to 1 and to prove that any Schubert class σ of degree c_1 is annihilated by a polynomial P with real roots.

For this, let us denote by H the subspace of degree c_1 classes in $QH^*(X, \mathbb{Z})_{q=1}$. To prove the above statement, it is enough to prove that the multiplication by σ in H is annihilated by a polynomial P with real roots. Indeed, if it is so, then we have $P(\sigma) \cdot 1 = 0$ because $1 \in H$ and thus $P(\sigma) = 0$. To prove that the multiplication by σ is annihilated by such a polynomial, we prove that it is an adjoint endomorphism for some positive definite real quadratic form on H. The form is given by the quantum cohomology using the formula

$$(a, b) = \text{coefficient of } q^2 \text{ in the product } a * b.$$

This pairing is clearly symmetric and the easy computation

$$(\sigma * a, b) = \text{coefficient of } q^2 \text{ in the product } \sigma * a * b = (a, \sigma * b)$$

proves that the multiplication by σ is adjoint for that form. For α a long simple root and ϖ_α the associated fundamental weight, using a formula in [**22**, Proposition 11.2], we get the equality

$$[\mathrm{pt}] * \sigma_{-\alpha} = \langle \varpi_\alpha, \Theta^\vee \rangle q^2 \sigma_\alpha.$$

This formula implies, after some computations, that the quantum Schubert classes of degree c_1 form an orthogonal basis and that the form is positive definite. □

We now state that the involution ι is not as simple as in the minuscule or cominuscule cases. In particular, we have the following.

THEOREM 7. *For X an adjoint non-coadjoint rational homogeneous space, there exists a Schubert class σ such that $\iota(\sigma)$ is not a multiple of a Schubert class.*

The proof of this result is rather involved. We produce, using a case-by-case analysis, a Schubert class σ as in the theorem. As this result is mainly a negative result, we will not give here the detailed proof and refer to [**14**, Theorem 6] for the details. We, however, give an example of such a class in the next remark.

REMARK 6.6. Let us compute explicitly in an example the element $\iota(\sigma)$ with σ as in the previous theorem. Assume $X = \mathbb{G}_Q(2, 7)$. We have $c_1 = 4$. In that case, we have $\sigma = \sigma_2$ with the notation of Proposition 6.2. Recall also that there is a unique other Schubert class of degree 2 denoted by τ_2. We have $\sigma^2 = 2\sigma_{-\alpha_{\mathrm{ad}}}$ and easy computations give the following equation:

$$\sigma^2(\sigma^4 - 4\sigma^2 - 16) = 0.$$

Therefore, in an identification of $H^*(\mathbb{G}_Q(2,7))_{q=1}$ with $\mathbb{R}^p \oplus \mathbb{C}^q$, the values of σ^2 are 0, $2(1+\sqrt{5})$ and $2(1-\sqrt{5})$. In particular, we see that the polynomial

$$Q(x) = \frac{x(x^2-2)}{2\sqrt{5}}$$

maps the square roots of the values of σ^2 to their complex conjugate, and therefore

$$\iota(\sigma) = \frac{\sigma^3 - 2\sigma q}{2q^2\sqrt{5}} = \frac{1}{q^2\sqrt{5}}(2\ell + 2q\tau_2 - q\sigma_2),$$

where ℓ is the class of a line.

7. *The adjoint variety of type A_n*

In this section we deal with the adjoint variety X of type A_n, namely, the incidence variety between points and hyperplanes in $\mathbb{P}^n$. For many aspects the results proved in the previous sections are still valid. However, as the Picard group of X is $\mathbb{Z}^2$, some of the results change. We explain in this section which of the results proved remain valid and which changes are required for the others. We only state the results and refer to [**14**, Section 7] for the proof.

Fact 7.1. (i) *The parametrization of Schubert classes by roots in Section* 2 *remains valid. In particular Proposition 2.9 is valid.*

(ii) *As the Picard group is $\mathbb{Z}^2$, there are two classes of degree* 1 *that we shall denote by h_1 and h_2, and there are two quantum parameters q_1 and q_2. Therefore, the parametrization of quantum monomials by roots of the affine root system fails.*

(iii) *The Chevalley and quantum Chevalley formulas also fail. However, setting $h = h_1 + h_2$ and $q = q_1 + q_2$, the formula in Theorem 3 is true.*

(iv) *The formula for affine symmetries fails by lack of a description of the quantum monomials.*

(v) *The description of ϖ-minuscule elements is valid as well as the fact that $d_{\max} = 2$.*

It is easy to derive an explicit form of the quantum Chevalley formula. This formula completely determines the quantum cohomology ring as the degree 1 classes are generators. We get the following proposition.

Proposition 7.2. *The quantum cohomology ring $QH^*(X,\mathbb{Q})$ is generated by h_1 and h_2 and the quantum parameters q_1 and q_2. It is the quotient of $\mathbb{Q}[h_1,h_2,q_1,q_2]$ by the relations*

$$\sum_{k=0}^{n} h_1^k(-h_2)^{n-k} = q_1 + (-1)^n q_2 \quad \text{and} \quad h_1^{n+1} = q_1(h_1+h_2).$$

The quantum cohomology ring is semi-simple for the quantum parameters q_1 and q_2 satisfying

$$q_1 \neq 0,\ q_2 \neq 0 \quad \textit{and} \quad q_1 + (-1)^n q_2 \neq 0.$$

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