Elliptic curves on the spinor varieties

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Abstract

In this article, we prove the irreducibility of the scheme of morphisms, of degree large enough, from a smooth elliptic curve to the spinor varieties. We give an explicit bound on the degree.

Introduction

We work over the field $\mathbb{C}$ of complex numbers. Let $\mathbb{C}^{2n}$ be endowed with a non-degenerate quadratic form $q$ and consider the orthogonal Grassmann variety $G_q(n, 2n)$ consisting of maximal isotropic subspaces in $\mathbb{C}^{2n}$. This variety has two isomorphic connected components. Let $X$ be any of these two components, the variety $X$ is called the spinor variety associated to $q$. It is a smooth rational homogeneous space of dimension $N = \frac{1}{2}n(n - 1)$. It is well known that the Picard group $\text{Pic}(X)$ is $\mathbb{Z}$ and is generated by a very ample line bundle. The degrees of the curves in $X$ will be computed with respect to this line bundle.

Let $C$ be a smooth elliptic curve, let $\alpha \in A_1(X)$ be a 1-cycle class in $X$ and denote by $\text{Hom}_\alpha(C, X)$ the scheme parametrising morphisms $f : C \to X$ from $C$ to $X$ such that $f_*[C] = \alpha$. Such a morphism is said to be of class $\alpha$. Our main result is the following.

**Theorem 0.1** Let $d$ be the degree of $\alpha$ and assume that the inequality $d \geq n - 1$ holds. Then the scheme $\text{Hom}_\alpha(C, X)$ is irreducible of dimension

$$\int_{\alpha} c_1(X) = 2(n - 1)d.$$

Apart from its own interest, this study was motivated by a question of D. Markushevich. Indeed, in a paper in collaboration with A. Iliev [IlMa07], they prove that the moduli space $\mathcal{M}_{X_12}(2, 1, 6)$ of rank 2 vector bundles with Chern classes $(1, 6)$ on the Fano threefold $X_{12}$ of index 1 and degree 12 can be identified via Serre’s construction with the family of elliptic curves of degree 6 on $X_{12}$. As $X_{12}$ is obtained from the above spinor variety of dimension 10 (in this case $n = 5$), the above result for $(n, d) = (5, 6)$ and a monodromy argument enables them to prove the irreducibility of $\mathcal{M}_{X_{12}}(2, 1, 6)$.

**Remark 0.2** A. Bruguières [Bru87] proved a similar statement for morphisms from elliptic curve to Grassmann varieties and E. Ballico [Bal89] proved similar statements for low genus curves on quadrics.

Using different techniques, we obtain, in a joint work with B. Pasquier in [PaPe11], another proof of Theorem 0.1 as well as irreducibility results for the scheme of morphisms from elliptic curves to some other homogeneous spaces. I expect that the technique used in this paper can be

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used to deal with cases which we could not solve in [PaPe11] like the Lagrangian Grassmann variety $\mathbb{G}_\omega(n,2n)$ of maximal isotropic subspaces in $\mathbb{C}^{2n}$ endowed with $\omega$ a non degenerate symplectic form.

To prove Theorem 0.1, we use the Bott-Samelson resolution $\pi_{F_\bullet}: \tilde{X}_{F_\bullet} \to X$. This is a proper birational morphism which depends on the choice of a complete flag $F_\bullet$ (see Section 1 and Subsection 2.1). The idea of the proof is to lift a morphism $f: C \to X$ to a morphism $\tilde{f}: C \to \tilde{X}_{F_\bullet}$. We then study the scheme of morphisms $\text{Hom}_{\tilde{C}}(C, \tilde{X}_{F_\bullet})$ for some special classes $\tilde{\alpha}$ such that $\pi_{F_\bullet*}\tilde{\alpha} = \alpha$. Since the variety $\tilde{X}_{F_\bullet}$ can be realised as a tower of $\mathbb{P}^1$-fibrations, we study $\text{Hom}_{\tilde{C}}(C, \tilde{X}_{F_\bullet})$ by induction along these fibrations. We in particular prove an irreducibility result on the scheme of morphisms for $\mathbb{P}^1$-fibrations (see Proposition 1.1) which is a generalisation to the case of elliptic curves of results proved in [Per02] and [Per05] for rational curves. To prove this result we need a strong positivity assumption on 1-cycle classes. We get from this the irreducibility of the scheme $\text{Hom}_{\tilde{C}}(C, \tilde{X}_{F_\bullet})$ for $\tilde{\alpha}$ in a subset of $A_1(\tilde{X}_{F_\bullet})$ denoted by $A_1^+(\tilde{X}_{F_\bullet})$ (see Corollary 1.5). The next step is to prove that for almost all morphisms $f \in \text{Hom}_{\tilde{C}}(C, X)$ we can choose the flag $F_\bullet$ in special position so that $f$ lifts to a morphism $\tilde{f}: C \to \tilde{X}_{F_\bullet}$ of class $\tilde{\alpha} \in A_1^+(\tilde{X}_{F_\bullet})$ (Proposition 2.4 and Proposition 3.1). We conclude by letting $F_\bullet$ vary.

As a consequence of Theorem 0.1, we can prove (see [PaPe11, Corollary 3.9]) the following result (we will not reproduce the proof here since it would be a verbatim of loc. cit.).

**Corollary 0.3** For $d \in [2, n]$, the scheme $\text{Hom}_{\tilde{C}}(C, X)$ is irreducible of dimension

$$2(n-1)d + \frac{1}{2}(n-d)(n-d-1).$$

**Convention** Let $G$ be the Spin group Spin($q$) associated to $q$. It is the (unique up to isomorphism) connected and simply connected algebraic group of Lie type $D_n$. Let $P$ be stabiliser of a maximal isotropic subspace in $\mathbb{C}^{2n}$ (recall that such a subspace is of dimension $n$). We shall denote by $\alpha_1, \cdots, \alpha_n$ the simple roots of $G$ with notation as in [Bou54]. In particular, the subgroup $P$ is a maximal parabolic subgroup of $G$ and we may assume that $P$ is associated to the simple root $\alpha_n$. The variety $X$ is isomorphic to $G/P$.

## 1 Elliptic curves on the Bott-Samelson resolution

In this section, we recall the basic properties of the Bott-Samelson resolution that we shall need and study the elliptic curves on it. We refer to [Dem74] or [BrKu05, Chapter 2] for more details on this construction. We shall give an equivalent construction due to P. Magyar [Mag98] in section 2.

### 1.1 Choice of a reduced expression

Let $W$ and $W_P$ be the Weyl groups of $G$ and $P$ respectively. The quotient $W/W_P$ has a nice set $W^P$ of representatives in $W$: the elements $w$ with minimal length in their class $wW_P$. Let $w_P$ be the element of maximal length in $W^P$. Since $X$ is minuscule, the element $w_P$ has a unique reduced expression modulo commuting relations (see for example [Ste97]):

$$w_P = s_{\beta_1} \cdots s_{\beta_N}$$

where the roots $\beta_i$ are simple roots.

Performing some commuting relations, we can choose the simple roots $(\beta_i)_{i \in [1,N]}$ as follows (recall that we number the simple roots $(\alpha_k)_{k \in [1,n]}$ as in [Bou54, Tables]). For each integer $k \in [1,n]$ we set

$$a_k = \frac{(k-1)(2n-k)}{2}$$

$$2$$
and define a partition of $[1, N]$ into sub-intervals $[a_k + 1, a_{k+1}]$ for $k \in [1, n - 1]$ (note that we have the equality $a_{k+1} = a_k + n - k$). For $i = a_k + j$ with $j \in [1, n - k]$, we set

$$\beta_i = \begin{cases} 
\alpha_{n-j} & \text{if } j \geq 2 \\
\alpha_n & \text{if } j = 1 \text{ and } n - k \text{ is odd} \\
\alpha_{n-1} & \text{if } j = 1 \text{ et } n - k \text{ is even.}
\end{cases}$$

This reduced expression can be represented using a quiver $Q$ which we already introduced in [Per07] and [ChMaPe08]. See a picture of the quiver $Q$ in the appendix.

### 1.2 Bott-Samelson resolution

Choose a complete isotropic flag $F_* = (F_1 \subset F_2 \subset \cdots \subset F_{n-2} \subset F_{n-1} = F_{n-1}^\perp, F_n = F_n^\perp \subset F_{n-2}^\perp \subset \cdots \subset F_2^\perp \subset F_1^\perp)$ where $F_i$ is an isotropic vector subspace of dimension $i$ in $\mathbb{C}^{2n}$ except for $F_{n-1}$ where $\dim F_{n-1} = n$. To the above reduced expression and the complete isotropic flag $F_*$ we may associate a variety $\tilde{X}_{F_*}$ called the Bott-Samelson variety. This variety can be described as a sequence of $\mathbb{P}^1$-fibrations:

$$\tilde{X}_{F_*} = X_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \cong \text{Spec}(k)$$

where each $f_i$ is a $\mathbb{P}^1$-fibration with a section $\sigma_i : X_{i-1} \to X_i$. We denote by $\xi_i$ the class of the divisor $D_i = \sigma_i(X_{i-1})$ in the Chow ring $A^*(X_i)$ and by $T_i$ the relative tangent bundle of the fibration $f_i$. By abuse of notation, we shall again denote by the same $\xi_i$ and $T_i$ the pull-backs of $\xi_i$ and $T_i$ in $\tilde{X}_{F_*}$.

### 1.3 Fibrations

We prove in this subsection, under positivity conditions on the cohomology class, that if the moduli space of elliptic curves in a variety $Y$ is irreducible, then the same is true for the moduli spaces of elliptic curves in $X$ where $X \to Y$ is a $\mathbb{P}^1$-fibration. A similar result was proved in [Per02, Proposition 4] for rational curves for which a much weaker positivity assumption was needed.

**Proposition 1.1** Let $C$ be a smooth elliptic curve. Let $\varphi : X \to Y$ be a $\mathbb{P}^1$-bundle with a section $\sigma : Y \to X$ and let $\tilde{\alpha} \in A_1(X)$. Let $T_\varphi$ be the relative tangent bundle and $\xi$ the class of the divisor $\sigma(Y)$. Assume that the conditions

$$\tilde{\alpha} \cdot \xi \geq 0, \tilde{\alpha} \cdot (T_\varphi - \xi) > 0 \text{ and } (\tilde{\alpha} \cdot \xi, \tilde{\alpha} \cdot (T_\varphi - \xi)) \neq (1, 1)$$

hold.

If $\text{Hom}_{\varphi, \tilde{\alpha}}(C, Y)$ is irreducible, then so is $\text{Hom}_{\tilde{\alpha}}(C, X)$ and we have the dimension equality

$$\dim(\text{Hom}_{\tilde{\alpha}}(C, X)) = \dim(\text{Hom}_{\varphi, \tilde{\alpha}}(C, Y)) + \tilde{\alpha} \cdot T_\varphi.$$

**Remark 1.2** For $(\tilde{\alpha} \cdot \xi, \tilde{\alpha} \cdot (T_\varphi - \xi)) = (1, 1)$ and if the scheme of morphisms $\text{Hom}_{\tilde{\alpha}}(C, X)$ is irreducible, we get that the scheme of morphisms $\text{Hom}_{\tilde{\alpha}}(C, X)$ is either empty or irreducible of dimension $\dim(\text{Hom}_{\tilde{\alpha}}(C, X)) = \dim(\text{Hom}_{\varphi, \tilde{\alpha}}(C, Y)) + \tilde{\alpha} \cdot T_\varphi$. The emptiness comes from the fact that there is no morphism of degree 1 from a smooth elliptic $C$ curve to $\mathbb{P}^1$.

**Proof.** Let $E$ be a rank two vector bundle on $Y$ such that $X = \mathbb{P}_Y(E)$. The section $\sigma$ is given by an injective map of vector bundles $N \to E$ where $N$ is an invertible sheaf on $Y$. Let $L$ be the cokernel of this map, it is also an invertible sheaf on $Y$. In this proof we shall use classical results
on vector bundles on elliptic curves, especially Serre duality and the fact that the canonical sheaf is trivial. For more on vector bundles on elliptic curves we refer to [Ati57].

Composition of morphisms defines a morphism \( \Phi : \text{Hom}_\mathbb{C}(C, X) \to \text{Hom}_{\mathbb{C}, \widetilde{\alpha}}(C, Y) \). We first study the fiber \( \Phi^{-1}(f) \) over a map \( f : C \to Y \). An element in this fiber is given by a lift of \( f \) i.e. an injective map of vector bundles \( K \to f^*E \) of vector bundles on \( C \) where \( K \) is an invertible sheaf.

Let \( M \) be the cokernel of this map which is again an invertible sheaf on \( C \).

One can easily check that in our situation, the relative tangent sheaf is \( \Lambda^2 E^\vee \otimes \mathcal{O}_{\mathbb{P}_Y(E)}(2) \) therefore we have the equality \( 2 \deg(M) = \deg(f^*E) = \alpha \cdot T_\varphi \). An element in the fiber is thus given by an invertible sheaf \( M \) on \( C \) with degree \( d = \frac{1}{2}(\deg(f^*E) + \alpha \cdot T_\varphi) \) and a surjective map in \( \mathbb{P}(\text{Hom}(f^*E, M)) \).

Note that we have the equality \( \alpha \cdot \xi = \deg(f^*L) - \deg(K) = \deg(M) - \deg(f^*N) \). Indeed the intersection of the lifted curve defined by \( K \to f^*E \) and the divisor \( \sigma(Y) \) is given by the locus where the composition \( K \to f^*E \to f^*L \) vanishes. We deduce from this relation the equality \( \alpha \cdot (T_\varphi - \xi) = \deg(M) - \deg(f^*L) \).

Note also that the group \( \text{Hom}(f^*E, M) \) is isomorphic to \( \text{Hom}(f^*N, M) \oplus \text{Hom}(f^*L, M) \). Indeed the group \( \text{Ext}^1(f^*L, M) = H^1(C, M \otimes (f^*L)^\vee) \) vanishes since \( M \otimes (f^*L)^\vee \) is of positive degree \( \alpha \cdot (T_\varphi - \xi) \) and \( C \) is an elliptic curve (use Serre duality for example). We discuss two cases.

If \( \alpha \cdot \xi > 0 \) then the two groups \( \text{Ext}^1(f^*L, M) = H^1(C, M \otimes (f^*L)^\vee) \) and \( \text{Ext}^1(f^*N, M) = H^1(C, M \otimes (f^*N)^\vee) \) both vanish since \( M \otimes (f^*L)^\vee \) and \( M \otimes (f^*N)^\vee \) are of positive degrees \( \alpha \cdot (T_\varphi - \xi) \) and \( \alpha \cdot \xi \).

The dimension of \( \text{Hom}(f^*E, M) \) is therefore \( \alpha \cdot T_\varphi \) and does not depend on \( f \). Furthermore, we claim that for a general invertible sheaf \( M \) with degree as above there exist surjections \( f^*E \to M \) as soon as we do not have

\[
f^*E \simeq f^*N \oplus f^*L, \quad f^*L \simeq f^*N \quad \text{and} \quad (\alpha \cdot \xi, \alpha \cdot (T_\varphi - \xi)) = (1, 1).
\]

This last case corresponds to the fact that the pull-back of the \( \mathbb{P}^1 \)-bundle \( \varphi : X \to Y \) to \( E \) is trivial. In that case and for the above conditions the sections are given by degree 1 morphisms \( C \to \mathbb{P}^1 \) and do not exist.

Let us give a short proof of the above claim. Since the surjectivity condition is open, it is enough to produce a line bundle \( M' \) of the correct degree with the desired surjectivity condition. If \( f^*E \) is the direct sum of \( f^*L \) and \( f^*N \), then taking two general morphisms \( f^*L \to M \) and \( f^*N \to M \) (we can choose them independent if \( L = N \) since \( (\alpha \cdot \xi, \alpha \cdot (T_\varphi - \xi)) \neq (1, 1) \) we get a morphism \( f^*E \to M \) which is surjective. If the extension is non trivial, then choose \( A \) a line bundle of degree \( \deg(f^*L) + \deg(f^*N) - \deg(M) \) and consider a non trivial map \( A \to f^*L \). Call \( Q \) its cokernel. Since \( \deg(A) < \deg(f^*N) \), we have the vanishing \( \text{Ext}^1(A, f^*N) = 0 \) therefore a surjective map \( \text{Ext}^1(Q, f^*N) \to \text{Ext}^1(L, N) \). Pick an extension lifting the extension corresponding to \( f^*E \), we get an exact diagram

\[
\begin{array}{ccccccccc}
0 & \to & f^*N & \to & f^*E & \to & f^*L & \to & 0 \\
& & & \downarrow s & \downarrow & & \downarrow & & \\
0 & \to & f^*N & \to & M' & \to & Q & \to & 0.
\end{array}
\]

The map \( s : f^*E \to M' \) is surjective while \( M' \) has the same degree as \( M \), the claim follows.

With our assumptions, the fiber is non empty and is an open subset of \( \mathbb{P}(\text{Hom}(f^*E, M)) \times \text{Pic}_d(C) \). This implies that the fibration \( \Phi \) is an open subset of the product of \( \text{Pic}_d(C) \) with a projective bundle of relative dimension \( \alpha \cdot T_\varphi - 1 \) above \( \text{Hom}_{\mathbb{C}, \widetilde{\alpha}}(C, Y) \). The result follows.
If $\tilde{\alpha} \cdot \xi = 0$, then if $M \neq f^*N$ we have $\text{Hom}(f^*N,M) = 0$ and any map $f^*E \to M$ factorises through $f^*L$ and is never surjective because $\text{deg}(M) - \text{deg}(f^*L) = \tilde{\alpha} \cdot (T_\varphi - \xi) > 0$. Therefore any pair $(M, p : f^*E \to M)$ of the fiber satisfies $M \simeq f^*N$. In that case, because of the equality $\text{deg}(M) - \text{deg}(f^*L) = \tilde{\alpha} \cdot (T_\varphi - \xi) > 0$, we have $\text{Ext}^1(f^*L, f^*N) = H^1(C, M \otimes (f^*L)^\vee) = 0$ thus the sheaf $f^*E$ is isomorphic to $M \oplus f^*L$. We have an isomorphism $\text{Hom}(f^*E, M) \simeq \text{Hom}(f^*N, M) \oplus \text{Hom}(f^*L, M)$. The dimension of $\text{Hom}(f^*E, M)$ is $\tilde{\alpha} \cdot T_\varphi + 1$ and does not depend on $f$. The fiber is therefore a non empty (there is always a surjective morphism in this case since $f^*N \simeq M$) open subset of $\mathbb{P}(\text{Hom}(f^*E, M))$. We therefore get an open subset of a projective bundle of relative dimension $\tilde{\alpha} \cdot T_\varphi$ above $\text{Hom}_{\mathbb{F}_q}(C, Y)$.

\section{Elliptic curves on the Bott-Samelson resolution}

We want to apply the previous result to the Bott-Samelson resolution $\tilde{X}_{F*}$. For this we need some positivity results on the classes $\tilde{\alpha}$ of curves in $\tilde{X}_{F*}$.

Remark 1.3 (i) Recall our definition of $T_i$ in subsection 1.2. A classical computation, see for example [Per05, Corollary 3.8], gives the following formula:

$$[T_i] = \sum_{k=1}^{i} (\gamma_k^\vee, \gamma_i) \xi_k$$

where $[T_i]$ is the class of $T_i$ and the roots $\gamma_i$ are defined by $\gamma_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$.

(ii) We also proved in [Per05, Corollary 2.16] that for a minuscule variety (recall that $X$ is a minuscule variety), we have for $k \neq i$ the inclusion $\langle \gamma_k^\vee, \gamma_i \rangle \in \{0, 1\}$. In the next lemma we explicitly compute some of these values.

Lemma 1.4 (i) Assume that $i = a_k + j$ for $k \in [2, n-1]$ and $j \in [1, n-k]$, then we have the equality $\langle \gamma_k^\vee, \gamma_i \rangle = 1$.

(ii) For any $i \neq j$ in $[1, n-1]$, we have the equality $\langle \gamma_j^\vee, \gamma_i \rangle = 1$.

Proof. (i) Cancelling some simple reflections, we get the equality

$$\langle \gamma_k^\vee, \gamma_i \rangle = \langle \beta_k^\vee, s_{\beta_k} \cdots s_{\beta_{i-1}}(\beta_i) \rangle.$$ 

On the other hand, a simple computation gives

$$s_{\beta_k} \cdots s_{\beta_{i-1}}(\beta_i) = \sum_{u=n-k-j+1}^{n-k} \alpha_u + 2 \sum_{u=n-k+1}^{n-2} \alpha_u + \alpha_{n-1} + \alpha_n$$

where the rightmost sum is empty for $k = 2$. The result follows since $\gamma_k = \beta_k = \alpha_{n-k}$.

(ii) In this case we have the equalities $\gamma_i = \beta_1 + \cdots + \beta_i$ and $\gamma_j = \beta_1 + \cdots + \beta_j$. The result follows.

Set $A^+_i(X_{F*}) = \{ \tilde{\alpha} \in A_1(X_{F*}) \mid \forall i \in [1, N], \tilde{\alpha} \cdot \xi_i \geq 0, \forall i \in [1, n-2], \tilde{\alpha} \cdot \xi_i > 0 \text{ and } \tilde{\alpha} \cdot T_1 \geq 4 \}$.

Corollary 1.5 (i) For $\tilde{\alpha} \in A^+_1(X_{F*})$, we have $(T_i - \xi_i) \cdot \tilde{\alpha} > \tilde{\alpha} \cdot \xi_i \geq 0$ for all $i \in [2, N]$.

(ii) In particular, for $\tilde{\alpha} \in A^+_1(X_{F*})$ the scheme $\text{Hom}_{\tilde{\alpha}}(C, X_{F*})$ is irreducible.
Proof. (i) Using Remark 1.3, and the fact that \( \langle \gamma^L_i, \gamma_i \rangle = 2 \), we have

\[
(T_i - \xi_i) \cdot \tilde{\alpha} = \xi_i \cdot \tilde{\alpha} + \sum_{k=1}^{i-1} \langle \gamma^L_k, \gamma_i \rangle \xi_k \cdot \tilde{\alpha}.
\]

Since the inequalities \( \xi_k \cdot \tilde{\alpha} \geq 0 \) and \( \langle \gamma^L_k, \gamma_i \rangle \geq 0 \) hold for any \( k \), we get, for any \( i \in [1, N] \) and any \( k \in [1, i - 1] \), the inequalities \( (T_i - \xi_i) \cdot \tilde{\alpha} \geq \xi_i \cdot \tilde{\alpha} + \langle \gamma^L_k, \gamma_i \rangle \xi_k \cdot \tilde{\alpha} \).

For \( i \in [1, N] \), write \( i = a_k + j \) with \( k \in [1, n-1] \) and \( j \in [1, n-k] \). If \( k \geq 2 \) holds, then \( \langle \gamma^L_{k-1}, \gamma_i \rangle = 1 \) holds and we get \( (T_i - \xi_i) \cdot \tilde{\alpha} \geq \xi_{k-1} \cdot \tilde{\alpha} + \xi_i \cdot \tilde{\alpha} > \xi_i \cdot \tilde{\alpha} \). Otherwise we have \( k = 1 \) and therefore the equality \( \langle \gamma^L_1, \gamma_i \rangle = 1 \) holds for \( i \geq 2 \). We get \( (T_i - \xi_i) \cdot \tilde{\alpha} \geq \xi_1 \cdot \tilde{\alpha} + \xi_i \cdot \tilde{\alpha} > \xi_i \cdot \tilde{\alpha} \).

(ii) We can apply Proposition 1.1 by induction on the \( \mathbb{P}^1 \)-fibrations given in the definition of the Bott-Samelson variety. Note that we have \( T_i = 2\xi_i \) and that the condition \( \tilde{a} \cdot T_1 \geq 4 \) is equivalent to the imposing that any morphism \( f : C \to \tilde{X}_{F_\alpha} \) of class \( \tilde{\alpha} \), when projected to \( X_1 \simeq \mathbb{P}^1 \) is of degree \( \tilde{\alpha} \cdot \xi_1 \geq 2 \). This is a necessary condition for the existence of such a morphism. □

Remark 1.6 For any complete flag \( F_\alpha \), there is a proper birational morphism \( \pi_{F_\alpha} : \tilde{X}_{F_\alpha} \to X \) (for more details see Subsection 2.1). We shall prove in Section 2 that for a general \( f \) in \( \text{Hom}_{\alpha}(C, X) \), we can choose a complete flag \( F_\alpha \) such that \( f \) can be lifted to an element \( \tilde{f} \) in \( \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_\alpha}) \) with the property \( \tilde{\alpha} \in A^+_1(\tilde{X}_{F_\alpha}) \).

1.5 Dimension

In this subsection, we compute the dimension of the scheme \( \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_\alpha}) \) for \( \tilde{\alpha} \in A^+_1(\tilde{X}_{F_\alpha}) \). This scheme is irreducible by Corollary 1.5. Its dimension can be computed by induction using Proposition 1.1. We get the equality

\[
\dim \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_\alpha}) = \sum_{i=1}^{N} T_i \cdot \tilde{\alpha} = -K_{\tilde{X}_{F_\alpha}} \cdot \tilde{\alpha}.
\]

On the other hand we computed in [Per07] the canonical divisor of the Bott-Samelson resolution of any minuscule Schubert variety (here \( X \) is a minuscule homogeneous space thus a minuscule Schubert variety). We get the equality

\[
-K_{\tilde{X}_{F_\alpha}} = \sum_{i=1}^{N} (h(i) + 1) \xi_i
\]

where \( h(i) \) is the height of the vertex \( i \) in the quiver \( Q \) i.e. the length of the longest path from the vertex \( i \) to the bottom vertex \( N \) (for example we have \( h(N) = 1 \), \( h(N-1) = 2 \) or \( h(1) = 2n - 3 \)). Furthermore, we also proved in [Per07], that if \( L \) is the ample generator of the Picard group of \( X \), then its pull-back by \( \pi_{F_\alpha} : \tilde{X}_{F_\alpha} \to X \) can be computed as follows:

\[
\pi_{F_\alpha}^* L = \sum_{i=1}^{N} \xi_i
\]

We therefore get the equality

\[
-K_{\tilde{X}_{F_\alpha}} = (h(1) + 1) \pi_{F_\alpha}^* L - \sum_{i=1}^{N} (h(1) - h(i)) \xi_i = 2(n-1) \pi_{F_\alpha}^* L - \sum_{i=1}^{N} (h(1) - h(i)) \xi_i.
\]
For any class $\tilde{\alpha} \in A_1^+(\tilde{X}_{F_\bullet})$ with $(\pi_{F_\bullet})_*\tilde{\alpha} = \alpha$, denote by $d$ the degree of $\alpha$ with respect to $L$. We get the inequality

$$-K_{\tilde{X}_{F_\bullet}} \cdot \tilde{\alpha} = 2(n-1)d - \sum_{i=1}^{N} (h(1) - h(i))\xi_i \cdot \tilde{\alpha} \leq 2(n-1)d - \sum_{i=1}^{n-2} (h(1) - h(i)).$$

Since for $i \in [1, n-1]$ we have $h(i) = 2(n-1) - i$ (see the quiver $Q$ in the appendix), we finally get the inequality

$$\dim \text{Hom}_{C}(C, \tilde{X}_{F_\bullet}) = -K_{\tilde{X}_{F_\bullet}} \cdot \tilde{\alpha} \leq 2(n-1)d - \frac{(n-2)(n-3)}{2}$$

with equality if and only if $\tilde{\alpha} \cdot \xi_i = 1$ holds for any $i \in [2, n-2]$ and $\tilde{\alpha} \cdot \xi_i = 0$ holds for any $i \in [n-1, N]$.

**Definition 1.7** Assume that $\alpha = \pi_{F_\bullet}^*\tilde{\alpha}$ satisfies $d = \alpha \cdot L \geq n-1$. Let us denote by $\tilde{\alpha}_0$ the unique class $\tilde{\alpha}$ in $A_1(\tilde{X}_{F_\bullet})$ with $\pi_{F_\bullet}^*\tilde{\alpha} = \alpha$ satisfying the conditions

- $\tilde{\alpha} \cdot \xi_i = 1$ for $i \in [2, n-2]$ and
- $\tilde{\alpha} \cdot \xi_i = 0$ for $i \in [n-1, N]$.

**Remark 1.8** Note that the condition $d \geq n-1$ is necessary for the existence of $\tilde{\alpha}_0$ since we have the inequalities

$$d = \tilde{\alpha}_0 \cdot \pi_{F_\bullet}^*L = \sum_{i=1}^{n} \tilde{\alpha}_0 \cdot \xi_i \geq \tilde{\alpha}_0 \cdot \xi_1 + \tilde{\alpha}_0 \cdot \xi_2 + \cdots + \tilde{\alpha}_0 \cdot \xi_{n-2} = 2 + n - 3 = n - 1.$$

## 2 Choice of the flag $F_\bullet$

### 2.1 More on the Bott-Samelson resolution

In [Mag98], P. Magyar gave another description of the Bott-Samelson resolution as a configuration variety. We reinterpreted this construction using the quiver $Q$ in [Per07].

In our situation, the configuration variety is given as follows. For $i \in [1, N]$, the simple root $\beta_i$ defines an isotropic Grassmann variety which we shall denote by $G_{iso}(\beta_i, 2n)$. This variety is simply the quotient $G/P_{\beta_i}$, where $P_{\beta_i}$ is the maximal parabolic subgroup such that $-\beta_i$ is not a root of $P_{\beta_i}$. Let us denote by $\dim \beta_i$ the dimension of an element in $G_{iso}(\beta_i, 2n)$ seen as an isotropic subspace of $\mathbb{C}^{2N}$. For a simple root $\alpha_i$, we have $\dim \alpha_i = i$ for all $i \neq n-1$ and $\dim \alpha_{n-1} = n$. For any vertex $i \in [1, N]$ of the quiver $Q$, we choose an element $x_i$ in $G_{iso}(\beta_i, 2n)$. We get in this way a collection

$$(x_i)_{i \in [1, N]} \in \prod_{i=1}^{N} G_{iso}(\beta_i, 2n).$$

We impose some conditions on this collection. First we fix some notation. For $i \in [1, N]$, we denote by $\delta_1$, and if they exist by $\delta_2$ and $\delta_3$, the simple roots such that $\langle \beta_i', \delta_k \rangle = -1$ for $k \in [1, 3]$ (there are the simple roots adjacent to $\beta_i$ in the Dynkin diagram, there may be one, two or three such simple roots). If any, denote by $j_1$, $j_2$ and $j_3$ the vertices of the quiver $Q$ with an arrow pointing to the vertex $i$ and such that $\beta_{j_k} = \delta_k$ for $k \in \{1, 2, 3\}$. We define the conditions

$$(*)_i : \text{For any } k \in \{1, 2, 3\}, \text{ we have } \left\{ \begin{array}{ll} x_i \subset x_{j_k} & \text{if } \dim \beta_i < \dim \delta_k \\ x_{j_k} \subset x_i & \text{if } \dim \beta_i > \dim \delta_k. \end{array} \right.$$
If the vertex \( j_k \) does not exist, we replace in the above condition the subspace \( x_{j_k} \) by the unique subspace of the complete flag \( F_\bullet \) that lies in the isotropic Grassmann variety \( \mathbb{G}_{iso}(\delta_k, 2n) \).

The Bott-Samelson can then be described as a configuration variety by the equality

\[
\tilde{X}_F = \left\{ (x_i)_{i \in [1,N]} \in \prod_{i=1}^{N} \mathbb{G}_{iso}(\beta_i, 2n) \mid (x_i)_{i \in [1,N]} \text{ satisfies } (*)_i \text{ for all } i \in [1,N] \right\}.
\]

The morphism \( \pi_{F_\bullet} : \tilde{X}_F \to X \) is given by the projection \((x_i)_{i \in [1,N]} \to x_N \). Furthermore, the sequence of \( \mathbb{P}^1 \)-fibration used in the previous section can be explicitly described by the sequence of morphisms

\[
X_N \to X_{N-1} \cdots X_{i+1} \to X_i \to \cdots X_1 \to pt
\]

where \( X_i \) is the image of the morphism with source \( X_N \) defined by \((x_k)_{k \in [1,N]} \mapsto (x_k)_{k \in [1,i]} \).

**Example 2.1** We describe here the Bott-Samelson resolution for \( n = 3 \). Fix a complete isotropic flag \( F_\bullet = (F_1 \subset F_2 \subset F_3 \subset F_2^\perp \subset F_1^\perp) \) with \( F_i \) an isotropic subspace of \( \mathbb{C}^6 \) of dimension \( i \). The Bott-Samelson resolution is the variety described as follows. By convention, all subspaces are isotropic and the indices denote the dimension of the subspaces. Furthermore \( V_3 \) and \( V_3' \) are in the same connected component \( X \) of the isotropic Grassmann variety \( \mathbb{G}_q(3, 6) \) while \( W_3 \) is in the other connected component.

\[
\tilde{X}_F = \{(V_3, V_2, V_1, W_3, V_3', V_3') \mid F_2 \subset V_3, F_1 \subset V_2 \subset V_3, V_1 \subset V_2 \subset W_3, V_1 \subset V_2' \subset W_3, V_2' \subset V_3')\}.
\]

With the above notation we have \((x_i)_{i \in [1,6]} = (V_3, V_2, V_1, W_3, V_3', V_3')\). One easily checks that \( \tilde{X}_F \) can be realised as a sequence of \( \mathbb{P}^1 \) fibrations. The map \( \pi_{F_\bullet} : \tilde{X}_F \to X \) is defined by projection on the last factor. We easily recover the fact that \( \pi_{F_\bullet} \) is birational since for \( V_2' \) general, we have

\[
V_2' = V_2' \cap F_1^\perp, \quad W_3 = V_2' \oplus F_1, \quad V_1 = V_3' \cap F_1^\perp, \quad V_2 = V_1 \oplus F_1 \quad \text{and} \quad V_3 = V_1 \oplus F_2.
\]

It is easy to check that the morphism \( \pi_{F_\bullet} \) is birational. In particular, for \( x_N \) general in \( X \), there is a unique collection \((x_i)_{i \in [1,N]} \in \tilde{X}_F \). For example, if \( V \) is the unique element in \( F_\bullet \) contained in the isotropic Grassmann variety \( \mathbb{G}_{iso}(\beta_n, 2n) \) (we have the equality \( V = F_{n-1} \) for \( n \) even and \( V = F_n \) for \( n \) odd), then we have for \( x_N \) general the equality

\[
x_{n-1} = x_N \cap V.
\]

The element \( x_{n-1} \) is a line in \( V \) i.e. a point in \( \mathbb{P}(V) \). For \( x_N \) general again, get also have the equalities

\[
x_i = (x_N \cap V) + F_{i-1} = x_{n-1} + F_{i-1} \quad \text{for } i \in [1, n-2].
\]

The element \( x_i \) is a subspace of dimension \( i \) in \( V \). It is therefore also an element of the Grassmann variety \( \mathbb{G}(i, V) \).

**2.2 A projection from \( X \)**

The element \( x_{n-1} \) is well defined by the formula \( x_{n-1} = x_N \cap V \) as soon as \( x_N \) meets \( V \) in dimension 1 and not more. Denote by \( U \) the open subset of elements \( x_N \) in \( X \) such that \( \dim x_N \cap V = 1 \). An easy dimension count gives that the complement of \( U \) is in codimension 3 in \( X \). On \( U \) we may define a morphism \( p : U \to \mathbb{P}(V) \) by setting \( p(x_N) = x_N \cap V \). This type of morphism was studied in more generality in [Per02]. In particular the following fact holds (see [Per02, Proposition 5], see also [PaPe11, Corollary 1.5(6)]).
Fact 2.2 The morphism \( p : U \to \mathbb{P}(V) \) realises \( U \) as the vector bundle associated to the locally free sheaf \( \Lambda^2 (T_{\mathbb{P}(V)}(-1)) \).

We shall also consider the projection \( q \) from \( \tilde{X}_{F_*} \) to the \((n-1)\) first terms of the collection \( i.e. \) the projection

\[
q : \tilde{X}_{F_*} \to \prod_{i=1}^{n-1} G_{iso}(\beta_i, 2n)
\]

defined by \( q((x_i)_{i \in [1,N]}) = (x_i)_{i \in [1,n-1]} \). The image of the projection is the variety \( X_{n-1} \) defined above. We shall denote it by \( \mathbb{P}_{F_*}(V) \) since it is the Bott-Samelson resolution \( \pi' : \mathbb{P}_{F_*}(V) \to \mathbb{P}(V) \) of the projective space \( \mathbb{P}(V) \) using the complete flag in \( V \) obtained from \( F_* \) by intersection with \( V \). We have the following commuting diagram

\[
\begin{array}{ccc}
\tilde{X}_{F_*} & \xrightarrow{q} & \mathbb{P}_{F_*}(V) \\
\downarrow{\pi_{F_*}} & & \downarrow{\pi'} \\
U_0 & \xrightarrow{p} & \mathbb{P}(V) \\
\end{array}
\]

where \( U_0 \) is the Bruhat cell associated to the complete flag \( F_* \) (\( i.e. \) the dense orbit in \( X \) of the stabiliser of \( F_* \)). Above \( U_0 \) the morphism \( \pi_{F_*} \) is an isomorphism.

Remark 2.3 The classes \( q_* \xi_i \) for \( i \in [1, n-1] \) form a basis of \( A^1(\mathbb{P}_{F_*}(V)) \). To simplify notation, we shall still denote them by \( \xi_i \).

2.3 Choice of the flag

Fix a morphism \( f : C \to X \) (we will not use the assumption elliptic in this subsection). We want to construct a complete flag \( F_* \) with nice properties with respect to \( f \). We start by fixing \( V \) which will be the subspace \( F_{n-1} \) for \( n \) even or \( F_n \) for \( n \) odd of the flag \( F_* \). We choose \( V \) such that the curve \( f(C) \) is contained in \( U \). This is possible using the classical Kleiman-Bertini Theorem [Klei74] and the fact that the complement of \( U \) is in codimension at least 2. The set of such subspaces \( V \) is open in the corresponding isotropic Grassmann variety.

Proposition 2.4 If the curve \( p \circ f(C) \) is not contained in a linear subspace of codimension 2, then we may find a complete flag \( F_* \) with \( F_{n-1} = V \) for \( n \) even and \( F_n = V \) for \( n \) odd such that \( f \) has a unique lift \( \tilde{f} : C \to \tilde{X}_{F_*} \) with \( \tilde{f}_*[C] \in A^1(\tilde{X}_{F_*}) \).

Remark 2.5 (i) First remark that a sufficient condition for the morphism \( f : C \to X \) to lift to \( \tilde{X}_{F_*} \), is that \( f(C) \) meets \( U_0 \) non trivially. Indeed, in that case \( f^{-1}(U_0) \) is open and dense in \( C \) therefore \( \tilde{f} \) is well defined on this open subset and since \( C \) is smooth and \( \tilde{X}_{F_*} \) projective we can extend this morphism to \( C \) itself.

(ii) Furthermore, remark that an element \( x \in U \) lies in \( U_0 \) if and only if its image \( p(x) \) under \( p \) is not contained in any subspace of the complete flag of \( V \) obtained by intersection with \( F_* \).

(iii) Finally remark that the datum of a complete \( F_* \) with \( F_{n-1} = V \) for \( n \) even and \( F_n = V \) for \( n \) odd is equivalent to the datum of a complete flag in \( V \).

We start with the following lemma.
Lemma 2.6 Let $g : C \to \mathbb{P}(V)$ be a morphism whose reduced image is not contained in any codimension 2 linear subspace. Then there exists a flag $F_\bullet$ with $F_{n-1} = V$ for $n$ even and $F_n = V$ for $n$ odd such that $g$ lifts to $\tilde{g} : C \to \widetilde{\mathbb{P}}(V)$ with $\tilde{g}_i[C] : \xi_i > 0$ for all $i \in [1, n-2]$ and $\tilde{g}_n[C] : \xi_{n-1} \geq 0$.

Proof. If we denote by $0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset V$ the complete flag of $V$ induced by $F_\bullet$ by intersection, we may describe the variety $\widetilde{\mathbb{P}}(V)$ as follows

$$\widetilde{\mathbb{P}}(V) = \left\{ (V_{n-i})_{i \in [1, n-1]} \in \prod_{i=1}^{n-1} G(n-i, V) \mid F_{i-1} \subset V_i \subset V_{i+1} \text{ for all } i \in [1, n-1] \right\}$$

with $V_n = V$. The morphism $\pi' : \widetilde{\mathbb{P}}(V) \to \mathbb{P}(V)$ is given by the projection on the last factor $(V_{n-i})_{i \in [1, n-1]} \mapsto V_1$. The divisor $D_i$ which is a representative of $\xi_i$ is given by the equation $V_{n-i} = F_{n-i}$.

We construct a collection of points $(P_i)_{i \in [1, n-1]}$ of $C$ by induction and deduce from this collection of points a collection of isotropic subspaces $F_i$ such that for all $i \in [1, n-1]$ the following conditions hold

$$\left\{ \begin{array}{l} g(P_i) \in \mathbb{P}(F_j) \text{ for } j > i \\ g(P_i) \notin \mathbb{P}(F_j) \end{array} \right.$$ \hspace{1cm} (1)

If such a collection $(P_i)_{i \in [1, n-1]}$ exists, then the point $g(P_{n-1})$ lies in the open subset where $\pi'$ is an isomorphism and we may lift $g$ to $\tilde{g} : C \to \widetilde{\mathbb{P}}(V)$ with the equality

$$\tilde{g}(P_{n-1}) = (g(P_{n-1}) + F_{n-2}, \cdots, g(P_{n-1}) + F_1, g(P_{n-1})).$$

In particular $\tilde{g}(P_{n-1})$ is not contained in any of the divisors $D_i$ defined above. We thus have the inequality $\tilde{g}_i[C] : \xi_i \geq 0$ for all $i \in [1, n-1]$. Furthermore, the point $\tilde{g}(P_i) = (V_{n-j}(P_j))_{j \in [1, n-1]}$ satisfies the condition $V_{i+1}(P_i) = g(P_i) + F_i = F_{i+1}$. We thus have the inclusion $\tilde{g}(P_i) \in D_{i+1}$ proving the inequality $\tilde{g}_i[C] : \xi_i > 0$ for all $i \in [1, n-2]$.

To construct the collections $(P_i)_{i \in [1, n-1]}$ and $(F_i)_{i \in [1, n-1]}$, we choose the points $P_i$ in general position in $C$. In particular the vector space spanned by $g(P_1), \cdots, g(P_i)$ is of dimension $i$ for all $i \in [1, n-1]$. We construct the subspaces $F_i$ by descending induction on $i$. The subspace $F_{n}$ is simply $V$. If $F_{i+1}$ is a fixed subspace containing $g(P_1), \cdots, g(P_i)$, we choose $F_i$ inside $F_{i+1}$ and containing $g(P_1), \cdots, g(P_{i-1})$ but not containing $g(P_i)$. We have a family of dimension 1 of such subspaces.

Remark 2.7 Note that from the above proof, for a given morphism $g : C \to \mathbb{P}(V)$, there is a family of dimension $2(n-1)$ of collections $(P_i)_{i \in [1, n-1]}$ and $(F_i)_{i \in [1, n-1]}$ satisfying the conditions (1) above. Therefore there exists a collection of dimension $2n - 3$ of complete flag satisfying the conclusion of the above lemma. Indeed, once the complete flag $F_\bullet$ is fixed, there is a finite number of choices for the points $P_i$ except for $x_{n-1}$ which is free.

Proof of Proposition 2.4. The above lemma tells us that there exists a complete flag $F_\bullet$ such that $g = p \circ f$ can be lifted in $\tilde{g} : C \to \widetilde{\mathbb{P}}(V)$ with $\tilde{g}_i[C] : \xi_i > 0$ for all $i \in [1, n-2]$. Furthermore, since $p \circ f(P_{n-1})$ is general with respect to the flag induced in $V$ by $F_\bullet$, the point $f(P_{n-1})$ lies in $U_0$. The morphism $f$ can therefore be lifted to $\tilde{f} : C \to \tilde{X}_F$ and we have $\tilde{g} = q \circ \tilde{f}$. Because $f$ meets $U_0$, we have for all $i \in [1, N]$, the inequality $\tilde{f}_i[C] : \xi_i > 0$. The inequalities $\tilde{g}_i[C] : \xi_i > 0$ for all $i \in [1, n-2]$ imply the inequalities $\tilde{f}_i[C] : \xi_i > 0$ for all $i \in [1, n-2]$. This finishes the proof of the proposition.

\hspace{1cm} □
3 Irreducibility

In this section we prove Theorem 0.1. Assume \( d \geq n - 1 \).

3.1 A dense open subset of morphisms

We start by proving that the closed subset of morphisms \( f : C \to X \) that do not satisfy the assumptions of Proposition 2.4 cannot contain any irreducible component of \( \text{Hom}_\alpha(C, X) \) for \( \alpha \) of degree \( d \) large enough. Recall that the morphism \( p : U \to \mathbb{P}(V) \) realises \( U \) as the vector bundle associated to the locally free sheaf \( \Lambda^2 T_{\mathbb{P}(V)}(-1) \).

**Proposition 3.1** Let \( \alpha \in A_1(X) \) be a class of degree \( d \geq n - 1 \). Then the closed subset \( M \) of \( \text{Hom}_{\alpha}(C, U) \) of morphisms \( f \) such that \( p \circ f(C) \) is contained in a linear subspace of codimension 2 never contains an irreducible component of \( \text{Hom}_{\alpha}(C, U) \).

**Proof.** Consider the morphism \( \text{Hom}_{\alpha}(C, U) \to \text{Hom}_{p^* \alpha}(C, \mathbb{P}(V)) \) induced by composition with \( p \). Note that to define \( p_* \) we consider the classes of 1-cycles as elements of the dual of the Picard group and use the transpose of \( p^* \). The image of \( M \) in \( \text{Hom}_{p^* \alpha}(C, \mathbb{P}(V)) \) is stratified by the families \( (M_k)_{k \in [2, n-1]} \) of morphisms \( f : C \to \mathbb{P}(V) \) whose image is contained in a unique linear subspace of codimension \( k \). An easy dimension count gives the equality \( \dim M_k = (n-k)d + k(n-k) \) (choose a subspace \( H \) of codimension \( k \), they form a family of dimension \( k(n-k) \) and then a morphism from \( C \) to \( H \), they form a family of dimension \( d(n-k) \)).

The fiber of the morphism \( \text{Hom}_{\alpha}(C, U) \to \text{Hom}_{p^* \alpha}(C, \mathbb{P}(V)) \) above \( f : C \to \mathbb{P}(V) \) is given by \( H^0(C, f^* \Lambda^2 T_{\mathbb{P}(V)}(-1)) \). If \( f \) is in \( M_k \), we have the equality

\[
f^* T_{\mathbb{P}(V)}(-1) = \mathcal{O}_C^k \oplus E
\]

where \( E \) is a locally free sheaf of rank \( n-1-k \) on \( C \), of degree \( d \), globally generated (since \( T_{\mathbb{P}(V)}(-1) \) is globally generated) and with no trivial factor (since the curve is contained in a linear subspace of codimension \( k \) and not more). Therefore \( f^* \Lambda^2 T_{\mathbb{P}(V)}(-1) \) is isomorphic to the direct sum

\[
\mathcal{O}_C^k \oplus E^k \oplus \Lambda^2 E.
\]

But since \( E \) is globally generated with no trivial factor and \( C \) is elliptic, we have the vanishing \( H^1(C, E) = 0 \). Here is a classical argument for this vanishing: if \( H^1(C, E) \) does not vanish then by Serre duality we get that \( H^0(C, E^\vee) \) does not vanish and we get a non trivial morphism \( E \to \mathcal{O}_C \). But since \( E \) is globally generated, the composition \( H^0(C, E) \otimes \mathcal{O}_C \to E \to \mathcal{O}_C \) is non trivial. This means that \( E \) has a trivial factor, a contradiction. We also have the vanishing \( H^1(C, \Lambda^2 E) = 0 \) (because the map \( H^0(C, E) \otimes E \to \Lambda^2 E \) is surjective). The cohomology group \( H^1(C, f^* \Lambda^2 T_{\mathbb{P}(V)}(-1)) \) is therefore of dimension \( \frac{k(k-1)}{2} \) and the cohomology group \( H^0(C, f^* \Lambda^2 T_{\mathbb{P}(V)}(-1)) \) is of dimension

\[
(n-2)d + \frac{k(k-1)}{2}.
\]

We get that the morphisms in \( \text{Hom}_{\alpha}(C, U) \) mapping to \( M_k \) form a subvariety of dimension

\[
(2n-k-2)d + k(n-k) + \frac{k(k-1)}{2}.
\]
Finally any irreducible component of $\text{Hom}_{\alpha}(C, U)$ is of dimension at least $2(n-1)d$. The inequality
\[
2(n-1)d > (2n-k-2)d + k(n-k) + \frac{k(k-1)}{2}
\]
for $d \geq n-1$ and $k \geq 2$ concludes the proof. \hfill $\square$

3.2 Incidence

Let $Y$ be the variety of complete isotropic flags. This variety is isomorphic to $G/B$ where $B$ is any Borel subgroup i.e. the stabiliser of a complete isotropic flag. We have seen that for any complete flag $F_* \in G/B$, there is a Bott-Samelson resolution $\pi_{F_*} : \tilde{X}_{F_*} \to X$. Consider the incidence variety $I$ defined as follows
\[
I \xrightarrow{pr_2} \text{Hom}_{\alpha}(C, X)
\]
where $I$ is the variety of pairs $(F_*, f) \in G/B \times \text{Hom}_{\alpha}(C, X)$ such that $f(C)$ is contained in the open subset $U$ defined by the flag $F_*$. The morphism $f$ therefore lifts to $\tilde{X}_{F_*}$ in a morphism $\tilde{f} : C \to \tilde{X}_{F_*}$ and we have $\tilde{f}_*[C] \in A_1^+(\tilde{X}_{F_*})$.

By Proposition 2.4 and Proposition 3.1, the morphism $pr_2 : I \to \text{Hom}_{\alpha}(C, X)$ is dominant as soon as $d \geq n-1$. Recall also (see Remark 2.7) that the dimension of the fiber of the morphism $pr_2$ above a general point is $2n-3 + \frac{n(n-1)}{2}$ (the last summand comes from the choice of $V$).

3.3 Irreducibility

By construction, we have a surjective morphism
\[
\bigoplus_{\tilde{\alpha} \in A_1^+(\tilde{X}_{F_*}), \pi_{F_*}^{-1}((\tilde{\alpha})) = \alpha} G/B \times \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_*}) \to I
\]
and by composition a dominant morphism
\[
\bigoplus_{\tilde{\alpha} \in A_1^+(\tilde{X}_{F_*}), \pi_{F_*}^{-1}((\tilde{\alpha})) = \alpha} G/B \times \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_*}) \to \text{Hom}_{\alpha}(C, X).
\]

We have seen in Subsection 1.5 that for any class of 1-cycle $\tilde{\alpha} \in A_1^+(\tilde{X}_{F_*})$, the dimension of $\text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_*})$ is at most
\[
2(n-1)d - \frac{(n-2)(n-3)}{2}
\]
with equality if and only of $\tilde{\alpha} = \tilde{\alpha}_0$. For $\tilde{\alpha} \in A_1^+(\tilde{X}_{F_*})$ such that $\pi_{F_*}^{-1}(\tilde{\alpha}) = \alpha$, the image of $G/B \times \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_*})$ in $\text{Hom}_{\alpha}(C, X)$ is therefore of dimension at most
\[
2(n-1)d - \frac{(n-2)(n-3)}{2} + \dim(G/B) - \dim(\text{pr}_2^{-1}(f)) \leq 2(n-1)d.
\]
But any irreducible component of $\text{Hom}_{\alpha}(C, X)$ is of dimension at least $2(n-1)d$. Therefore for $\tilde{\alpha} \neq \tilde{\alpha}_0$ the image of $G/B \times \text{Hom}_{\tilde{\alpha}}(C, \tilde{X}_{F_*})$ in $\text{Hom}_{\alpha}(C, X)$ does not contain any irreducible component. We therefore have a dominant morphism $G/B \times \text{Hom}_{\tilde{\alpha}_0}(C, \tilde{X}_{F_*}) \to \text{Hom}_{\alpha}(C, X)$. The scheme $\text{Hom}_{\tilde{\alpha}_0}(C, \tilde{X}_{F_*})$ being irreducible this concludes the proof of Theorem 0.1.
4 Appendix

We give here a picture of the quiver $Q$ obtained from the reduced expression defined in Subsection 1.1. This quiver slightly changes for $n$ odd or $n$ even.

The map $\beta$ from the set of vertices of $Q$ to the set of simple roots $S$ is given by the vertical projection. We do not draw the arrows on the edges: all arrows are going down.

References


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